Exam n° 3 – 1 hour 30 minutes

Warm-up exercises (10 points)

You are expected to provide some steps for those exercises. Little partial credit will be given for just writing the answer.

Exercise 1. Sense or no sense, that is the question. Justify briefly why the following statements are meaningful or meaningless.

1. Sense or No Sense : *The function* $f : \mathbb{R}^2 \to \mathbb{R}$ *admits an image of dimension* 2 *so it is bijective.*
2. Sense or No Sense : *Consider the linear application* $p : E \to E$ *such that* $p \circ (-p) = -p$. *Then* p *is a projection.*
3. Sense or No Sense : *The rank of the kernel is linearly independent so it is injective.*
4. Sense or No Sense : $x = o(1)$ *at* $x = 0$, *but* $1 = o(x)$ *at* $x = \infty$.

Solution.

1. No Sense. $\text{Im}(f)$ is a vector space of $\mathbb{R}$ so $\dim(\text{Im}(f)) \leq 1$ (cannot be more).
2. Sense. By linearity $p \circ (-p) = -(p \circ p)$ and $p \circ (-p) = -p$, so $-(p \circ p) = -p \iff p \circ p = p$, therefore p is a projection.
3. No Sense. Every word together is garbage (kernel has no rank, it cannot be linearly independent but the family generating is, and it is not a application). If someone dared write *Sense*, a death threat (aka draw a skull) should have been initiated.
4. Sense. $\dfrac{x}{1} \underset{x \to 0}{\to} 0$ and $\dfrac{1}{x} \underset{x \to \infty}{\to} 0$.

□

Exercise 2. Consider $s = (1, 0, 0, 1)$, $t = (0, 1, 1, 1)$, $u = (0, -1, -1, 1)$, $v = (2, 0, 0, 0)$, and $B = (s, t, u, v)$ a family of $\mathbb{R}^4$.

1. Show that B is linearly dependent and provide its rank.
2. Complete the linearly independent subfamily to form a basis of $\mathbb{R}^4$.

Solution.

— Several options here (but the questions leans towards using the linear combination of vectors). Let's find $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\alpha s + \beta t + \gamma u + \delta v = 0_{\mathbb{R}^4}$.

$$\begin{cases} \alpha + 2\delta = 0 \\ \beta - \gamma = 0 \\ \beta - \gamma = 0 \\ \alpha + \beta + \gamma = 0 \end{cases}$$

Removing L_3 and swapping $L_1 \leftrightarrow L_4$ we obain :

$$\begin{cases} \alpha+\beta+\gamma=0 \\ \beta-\gamma=0 \\ \alpha+2\delta=0 \end{cases} \quad \overset{L_3 \leftarrow L_3 - L_1}{\Leftrightarrow} \quad \begin{cases} \alpha+\beta+\gamma=0 \\ \beta-\gamma=0 \\ -\beta-\gamma+2\delta=0 \end{cases} \quad \overset{L_3 \leftarrow L_3 + L_2}{\Leftrightarrow}$$

$$\begin{cases} \alpha+\beta+\gamma=0 \\ -\beta-\gamma=0 \\ -2\gamma+2\delta=0 \end{cases} \iff \begin{cases} \alpha=-2\delta \\ \beta=\delta \\ \gamma=\delta \\ \delta=\delta \end{cases} \quad ,\delta\in\mathbb{R}$$

There are 3 equations for 4 unknowns, so 1 free parameter. The system admits non trivial solutions so B is linearly dependent and $\operatorname{rk}(B)=3$. In particular choosing $\delta=1$ leads to :

$$v=2s-t-u$$

so we can write $\operatorname{Span}(s,t,u,v)=\operatorname{Span}(s,t,u)$.

— We complete the family (s,t,u) with one vector from the canonical basis of $\mathbb{R}^4$. We choose e_3 (e_2 should also work).

□

Exercise 3. Consider $F=\operatorname{Span}(e_1,e_2)$, $G=\operatorname{Span}(e_4-e_1,e_3)$ two vector subspaces of $\mathbb{R}^4$, expressed using the canonical basis of $\mathbb{R}^4$.

1. Provide the dimension of F, G.
2. Compute $F\cap G$. Are F and G supplementary ?
3. Deduce a relation between F, G, and $\mathbb{R}^4$.

Solution. 1. (e_1,e_2) is a linearly independent family so $\dim F=2$. (e_4-e_1,e_3) is a linearly independent family so $\dim G=2$.

2. Let $u\in F\cap G$. Then $\exists\alpha,\beta,\gamma,\delta\in\mathbb{R}$ such that $u=\begin{bmatrix}\alpha\\ \beta\\ 0\\ 0\end{bmatrix}=\begin{bmatrix}-\gamma\\ 0\\ \delta\\ \gamma\end{bmatrix}\iff \alpha=\beta=\gamma=\delta=0$. So $F\cap G=\{0_{\mathbb{R}^4}\}$. F and G are 2 vectors spaces of $\mathbb{R}^4$ with $F\cap G=\{0_{\mathbb{R}^4}\}$, they are supplementary.
3. We conclude that $F\oplus G=\mathbb{R}^4$.

□

Exercise 4. Provide an equivalent at $x=0$ of the following function (justify your steps) and deduce the limit.

$$f(x)=\frac{\sin(2x)\ln(1+x)+x\sin(2x)\ln(1+x)}{x(x+1)}$$

Solution. We simplify f by factorizing $x+1$, and using common equivalents (and property of multiplying equivalents) :

$$f(x)=\underbrace{\frac{\sin(2x)}{2x}}_{\sim_0 1}\left[\underbrace{2\ln(1+x)}_{\sim_0 2x}\right]\sim_0 2x$$

Then $\lim_{x\to 0} f(x)=0$. □

Exercise 5. Find the following limit (justify your result).

$$\lim_{x\to+\infty}\left(\frac{x^2-1}{x^2}\right)^x$$

Solution. We write $\left(\frac{x^2-1}{x^2}\right)^x = \left(1-\frac{1}{x^2}\right)^x = e^{x\ln\left(1-\frac{1}{x^2}\right)}$. Using the fact that $\ln(1+h) =_0 h + o(h)$, we deduce by composition $x\ln\left(1-\frac{1}{x^2}\right) =_{+\infty} x\left(-\frac{1}{x^2}+o\left(\frac{1}{x^2}\right)\right) = -\frac{1}{x}+o\left(\frac{1}{x}\right)$. We conclude that $\left(\frac{x^2-1}{x^2}\right)^x \sim_{+\infty} e^{-1/x}$, therefore

$$\lim_{x\to+\infty}\left(\frac{x^2-1}{x^2}\right)^x = 1$$

□

Linear applications (4 points)

Exercise 6.
Consider the linear application $f:\mathbb{R}^3\to\mathbb{R}^4$ such that $f(x,y,z)=(x+y,-x-y,3x+3y,2z)$.

1. Write the associated matrix M relative to the canonical basis of $\mathbb{R}^3$ and $\mathbb{R}^4$.
2. Without computing neither the kernel or the image, determine the dimension of one of them (state explicitly which one you pick), and **deduce** the dimension of the other. Justify your answer.
3. Can f be injective ? Surjective ? Bijective ? Justify each answer.
4. Provide a basis K of $\ker(f)$, and a basis I of $\mathrm{Im}(f)$.
5. Complete K with vectors to form a basis B of $\mathbb{R}^3$, and write the associated matrix M' relative to B and the canonical basis of $\mathbb{R}^4$.

Solution. 1. $M = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

2. Option 1 : $(x,y,z)\in\ker(f) \iff \begin{cases} x+y-0 \\ -x-y=0 \\ 3x+3y=0 \\ 2z=0 \end{cases}$. The first the rows are proportional, we then have 3 unknowns and 2 equations, so 1 free parameter. The kernel is of dimension 1. The rank theorem gives us $\mathrm{rk}(f)=\dim\mathbb{R}^3-\dim\ker(f)=2$.
   Option 2 : The first the columns of M are the same, we deduce that $\mathrm{rk}(M)=2=\mathrm{rk}(f)$. The rank theorem gives us $\dim\ker(f)=\dim\mathbb{R}^3-\mathrm{rk}(f)=1$.

3. f cannot be injective ($\ker f\neq\{0_{\mathbb{R}^3}\}$), cannot be surjective ($\mathrm{rk}(f)<\dim\mathbb{R}^4$), therefore cannot be bijective.

4. $(x,y,z) \in \ker(f) \iff \begin{cases} x+y-0 \\ 2z=0 \end{cases} \iff \begin{cases} x+y-0 \\ x=-\lambda \\ y=\lambda \\ z=0 \end{cases}, \quad \lambda \in \mathbb{R}$. So $\ker(f) = \text{Span}\left(\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\right)$.

   We set $K = \left(\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\right)$ basis of $\ker(f)$ (generating and linearly independent).

   Using M and/or the definition of $\text{Im}(f)$ we have :

   $\text{Im}(f) = \text{Span}(f(e_1), f(e_2), f(e_3)) = \text{Span}\left(\begin{pmatrix} 1 \\ -1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \end{pmatrix}\right) = \text{Span}\left(\begin{pmatrix} 1 \\ -1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \end{pmatrix}\right)$.

   We set $I = \left(\begin{pmatrix} 1 \\ -1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \end{pmatrix}\right)$ basis of $\text{Im}(f)$ (generating and linearly independent).

5. We complete K with 2 vectors from the canonical basis of $\mathbb{R}^3$: $B = \left(\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)$ (the chosen family is linearly independent and of rank 3). Then $M' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (image of the first vector is 0, the other two have been already computed).

□

Limits and Comparisons (6 points)

Exercise 7.

Consider $f(x) = \dfrac{\cos(x)(\cos(x)-1)+\sin^2(x)}{x(1-e^{3x})}$.

1. Determine the domain of definition of f.
2. Provide an equivalent of f at 0 and deduce the limit when $x \to 0$.
3. Can f be extended by continuity to $\mathbb{R}$? Justify. If yes, provide the expression of $\tilde{f}$, the extension.
4. Behavior at $+\infty$:
   (a) Find g such that $f(x) \underset{+\infty}{=} O(g(x))$.
   (b) Compute the limit of f at $+\infty$.
   (c) Conclude whether f admits an asymptote and $+\infty$ (if yes, specify what kind).
5. Behavior at $-\infty$: proceed similarly as in previous question (comparison, limit, asymptote).

Solution. 1. We remark that $f(x) = \dfrac{1-\cos(x)}{x(1-e^{3x})}$, then $D_f = \mathbb{R}^*$.

2. Using usual equivalents and ratio properties : $1-\cos(x) \sim_0 \dfrac{x^2}{2}$, $1-e^{3x} \sim_0 -3x$ so $f(x) \sim_0 -\dfrac{\frac{x^2}{2}}{3x^2} = -\dfrac{1}{6}$. Therefore $\lim_{x\to 0} f(x) = -\dfrac{1}{6}$.

3. f can be extended by continuity to $\mathbb{R}$ via the following function $\tilde{f}(x) = \begin{cases} \dfrac{1-\cos(x)}{x(1-e^{3x})} & x \neq 0 \\ -\frac{1}{6} & x = 0 \end{cases}$.

4. Let us denote $\mathcal{V}^+$ the vicinity of $+\infty$.

   (a) $\forall x \in \mathcal{V}^+$, $|f(x)| \leq \dfrac{2}{x(e^{3x}-1)}$ so we can choose for example $g(x) = \dfrac{1}{x(e^{3x}-1)}$.

   (b) It turns out $g(x) \sim_0 x^{-1}e^{-3x} \underset{x\to+\infty}{\longrightarrow} 0$. Or similarly $g(x) = \dfrac{1}{x(e^{3x}-1)} = \underset{\to 0}{e^{-3x}} \underbrace{\dfrac{1}{x(1-e^{-3x})}}_{\to 0} \underset{x\to+\infty}{\longrightarrow} 0$. Therefore $\lim\limits_{x\to+\infty} f(x) = 0$.

   (c) We conclude that f admits a horizontal asymptote $y = 0$ at $+\infty$.

5. Let us denote $\mathcal{V}^-$ the vicinity of $-\infty$.

   (a) $\forall x \in \mathcal{V}^-$, $|f(x)| \leq \dfrac{2}{|x|(1-e^{3x})}$ so we can choose for example $h(x) = g(x)$.

   (b) It turns out $h(x) \sim_0 \dfrac{1}{x} \underset{x\to+\infty}{\longrightarrow} 0$. Therefore $\lim\limits_{x\to-\infty} f(x) = 0$.

   (c) We conclude that f admits a horizontal asymptote $y = 0$ at $-\infty$.

$\square$