Exam n° 3 – Solutions

- No documents, no calculators, no cell phones or electronic devices allowed.
- Take a deep breath before starting (everything is going to be ok!) and read entirely the exam before starting.[0]
- All exercises are independent, you can do them in the order that you'd like.
- Please start an exercise at the top of a page (for readability).
- Number single pages, or simply the booklets (*copies doubles*) if multiple : for example 1/3, 2/3, 3/3
- All your answers must be fully (but concisely) justified, unless noted otherwise.
- Redaction and presentation matter! For instance, write full sentences and make sure your 'x' and 'n' can be distinguished.
- Questions marked (*) are considered more challenging.
- **Respecting all of the above is part of the exam grade (0.5 points).** Provided rubric is indicative (changes may occur).

Fundamental exercises (11,5 points)

You are expected to provide some steps for those exercises. Little partial credit will be given for just writing the answer.

Exercise 1. Does it make sense or not? Justify briefly if the following statements use a valid reasoning or not, or if they are True or False.
If it's not a valid reasoning, simply indicate where is the problem (or what is missing).
If it's False, provide the correct answer (if applicable).

1. *Valid reasoning?* Consider $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x+y) = f(x) + f(y)$, $\forall (x,y) \in \mathbb{R}^2$. Then f is a linear application.
2. *Valid reasoning?* Consider $f \in \mathcal{L}(\mathbb{R}^n)$, such that $\forall x \in \mathbb{R}^n$ $f^4(x) = f(x)$. Since $f^4 = f^2 \circ f^2 = f$, then f is a projection.
3. *True or False?* Using equivalents we find that $\sin(x) - x \underset{0}{\sim} 0$.
4. *True or False?* If $F = \text{Span}((1,0))$ and $G = \text{Span}((-1,1))$ then $F \oplus G = \mathbb{R}^2$.

Solution.

1. Not valid : in order to be a linear application, f also needs to satisfy $f(\lambda x) = \lambda f(x)$ for all $x \in \mathbb{R}^2$, $\lambda \in \mathbb{R}$.
2. Not valid : in order to be a projection f needs to satisfy $f \circ f = f$. The fact that $f^2 \circ f^2 = f$ doesn't imply the projection property (the other way around works though).
3. False : never write a function equivalent to 0! We have $\sin(x) - x \underset{0}{\sim} -\frac{x^3}{3!}$ (next term in the TSE).
4. True : $F + G \subset \mathbb{R}^2$ and $F \cap G = \{0_{\mathbb{R}^2}\}$ so $F \oplus G = \mathbb{R}^2$.

□

Exercise 2. Solve $\cos(2x) = -1 + \sin(2x)$ over $\mathbb{R}$.

Solution.

$$\cos(2x) = -1 + \sin(2x) \iff 2\cos^2(x) - 1 = -1 + 2\cos(x)\sin(x) \iff \cos(x)\left(\cos(x) - \sin(x)\right) = 0$$

We conclude

$$\begin{cases} \cos(x) = 0 \\ \text{or} \\ \cos(x) = \sin(x) \end{cases} \iff \begin{cases} x = \dfrac{\pi}{2} + k\pi, & k \in \mathbb{Z} \\ \text{or} \\ x = \dfrac{\pi}{4} + k\pi, & k \in \mathbb{Z} \end{cases}$$

so we have $S = \{\frac{\pi}{4}, \frac{\pi}{2}\} + \pi\mathbb{Z}$. □

Exercise 3. Provide an equivalent of the following functions (justify your steps) and deduce their limit.

$$f(x) = \frac{x^2 \ln(\frac{1+x}{x}) + \ln(x^3)}{x^3(1-2x)} \quad \text{at } +\infty$$

$$g(x) = \frac{\ln(1-\tan(x))}{\sin(3x)} \quad \text{at } 0$$

Solution. For f :

— We have $\ln(\frac{1+x}{x}) = \ln(1 + \frac{1}{x}) \underset{\infty}{\sim} \frac{1}{x}$ so $x^2 \ln(\frac{1+x}{x}) \underset{\infty}{\sim} x$

— By comparative growth $\ln(x^3) = 3\ln(x) \underset{\infty}{=} o(x)$

— We conclude $x^2 \ln(\frac{1+x}{x}) + \ln(x^3) \underset{\infty}{\sim} x$

— We have $x^3(1-2x) \underset{\infty}{\sim} -2x^4$

— We find $f(x) \underset{\infty}{\sim} -\frac{1}{2x^3} \underset{x\to+\infty}{\longrightarrow} 0$

For g :

— We have $\tan(x) \underset{0}{\sim} x$ by composition $\ln(1-\tan(x)) \underset{0}{\sim} -x$

— We have $\sin(3x) \underset{0}{\sim} 3x$

— We find $g(x) \underset{\infty}{\sim} -\frac{1}{3} \iff \lim_{x\to 0} g(x) = -\frac{1}{3}$.

□

Exercise 4. Consider $s = (1,0,1)$, $t = (1,1,1)$, $u = (0,-2,0)$.

1. Compute the rank of the family (s,t,u).
2. Is this family a basis of $\mathbb{R}^3$? Justify your answer. In case it is not, complete the linearly independent subfamily to form a basis of $\mathbb{R}^3$. We will call this family B.
3. Consider $v = (x,y,z)$ expressed in the canonical basis of $\mathbb{R}^3$. Express $[v]_B$, that is v in the basis B.

Solution. 1. We find

$$\text{rk}\left(\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\-2\\0\end{pmatrix}\right) = \text{rk}\left(\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\-2\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right) \underset{C_3 \leftarrow C_3 - C_1}{=} \text{rk}\left(\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\-2\\0\end{pmatrix}, \begin{pmatrix}0\\-1\\0\end{pmatrix}\right) = 2$$

(we end up with a row echelon form after canceling the last column).

2. This family is not a basis of $\mathbb{R}^3$ since $\text{rk}(s,t,u) = 2 \neq 3 = \dim(\mathbb{R}^3)$. We keep t, u (for example) and complete the family with one vector of the canonical basis. We choose e_1 so that

$$\text{rk}(t,u,e_1) = \text{rk}\left(\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\-2\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}\right) = 3$$

(row echelon form) so it is a basis.

3. We express (t, u, e_1) in the canonical basis (e_1, e_2, e_3) : $t = e_1 + e_2 + e_3$, $u = -2e_2$ and $e_1 = e_1$. So we want to find (a, b, c) such that $[v]_B = (a, b, c)_B = (x, y, z)$. We have

$$(a, b, c)_B = (a + c)e_1 + (a - 2b)e_2 + ae_3 = xe_1 + ye_2 + ze_z \iff \begin{cases} a = x - z \\ b = \dfrac{z - y}{2} \\ a = z \end{cases}$$

so $[v]_B = (x - z, \dfrac{z - y}{2}, z)_B$.

□

Exercise 5. We want to determine the regularity of the function defined over $\mathbb{R}$ by

$$f(x) = \begin{cases} \sin(x^3) - x^3 & x \leq 0 \\ x^6 \ln(1 + x) & x > 0 \end{cases}$$

To that aim we define $f_1(x) = \sin(x^3) - x^3$ for $x \leq 0$, and $f_2(x) = x^6 \ln(1 + x)$ for $x \geq 0$.

1. What is the regularity of f_1, f_2 ?
2. Based on previous question provide their associated Taylor series expansions at $x = 0$, with at least one non-zeros term and both of the same order.
3. How can we deduce the TSEs of their respective derivatives at 0 ? From this find n, m such that $f_1^{(n)}(0) \neq 0$, $f_2^{(m)}(0) \neq 0$.
4. Deduce the class of the function f.

Solution. 1. $f_1 \in \mathcal{C}^\infty(\mathbb{R}_-)$, $f_2 \in \mathcal{C}^\infty(\mathbb{R}_+)$ by composition of $\mathcal{C}^\infty$ functions.

2. By composition we have $f_1(x) \underset{0}{=} -\dfrac{x^9}{3!} + o(x^9)$, and by product we find $f_2(x) \underset{0}{=} x^7 - \dfrac{x^8}{2} + \dfrac{x^9}{3} + o(x^9)$.

3. Due to the regularity of the functions f_1, f_2 the TSEs of their derivatives at 0 are the derivatives of the TSE found in previous question. We then find that

$$f_1^{(9)}(x) \underset{0}{=} \frac{9!}{3!} + o(1) \quad f_2^{(7)}(x) \underset{0}{=} 7! - \frac{8!}{2}x \frac{9!}{6}x^2 + o(x^2)$$

so $n = 9$, $m = 7$ and $f_1^{(9)}(0) = \dfrac{9!}{3!}$, $f_2^{(7)}(0) = 7!$.

4. We can check that $f_1^{(n)}(0) = 0 = f_2^{(n)}(0)$ for $n = 0, 1, \ldots, 6$, but $f_1^{(7)}(0) = 0 \neq f_2^{(7)}(0) = 7!$ so $f \in \mathcal{C}^6(\mathbb{R})$.

□

Linear applications (4 points)

Exercise 6.

Consider the linear application $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that $f(x, y, z) = (x - 2y + 3z, -6z - 2x + 4y, -3z - x)$.

1. Write the associated matrix M relative to the canonical basis of $\mathbb{R}^3$.

2. Without computing neither the kernel or the image, determine the dimension of one of them (state explicitly which one you pick), and **deduce** the dimension of the other. Justify your answer.
3. Can f be injective ? Surjective ? Bijective ? Justify each answer.
4. Provide a basis K of $\ker(f)$, and a basis I of $\mathrm{Im}(f)$. We expect proper justification here.
5. Show that $\ker(f) \oplus \mathrm{Im}(f) = \mathbb{R}^3$.
6. (*) Can we conclude that f is a projection ? Justify your answer.

Solution. 1.

$$M = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ -1 & 0 & -3 \end{bmatrix}$$

2. We see that 2 columns of M are proportional ($C_3 = 3C_1$) so $\mathrm{rk}(f) = 2$ and by Grassman we deduce $\dim\ker(f) = \dim(\mathbb{R}^3) - \mathrm{rk}(f) = 3 - 2 = 1$.
3. From previous question, f is not injective ($\dim\ker(f) = 1$) and since $f \in \mathcal{L}(\mathbb{R}^3)$ it's automatically not surjective and therefore not bijective.
4.

$$f(x,y,z) = 0_{\mathbb{R}^3} \Leftrightarrow \begin{pmatrix} x & y & z & |b \\ 1 & -2 & 3 & |0 \\ -2 & 4 & -6 & |0 \\ -1 & 0 & -3 & |0 \end{pmatrix} \overset{\substack{L_2 \leftarrow L_2 + 2L_1 \\ L_3 \leftarrow L_3 + L_1}}{\Leftrightarrow} \begin{pmatrix} x & y & z & |b \\ 1 & -2 & 3 & |0 \\ 0 & 0 & 0 & |0 \\ 0 & -2 & 0 & |0 \end{pmatrix} \Leftrightarrow \begin{cases} x = -3\alpha \\ y = 0 \\ z = \alpha \end{cases}, \alpha \in \mathbb{R}$$

so $\ker(f) = \mathrm{Span}\left(\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}\right)$. So $K = \left(\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}\right)$ is generating and L.I. so a basis of $\ker(f)$.

$\mathrm{Im}(f) = \mathrm{Span}\left(\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -6 \\ -3 \end{pmatrix}\right) = \mathrm{Span}\left(\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix}\right)$ (last vector being proportional to the first one). So $I = \left(\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix}\right)$ is generating and L.I. (row echelonned) to it is a basis of $\mathrm{Im}(f)$.

5. By definition $\dim(\ker(f)) + \dim(\mathrm{Im}(f)) = \dim(\mathbb{R}^3)$ and $\ker(f) + \mathrm{Im}(f) \subset \mathbb{R}$ (because $f \in \mathcal{L}(\mathbb{R}^3)$ otherwise not true !) therefore $\ker(f) \oplus \mathrm{Im}(f) = \mathbb{R}^3$.
6. Let us check that $f \circ f = f$. After tedious calculations we find that $f \circ f \neq f$ so it is not a projection. One can also use the matrix M directly and show that $M^2 \neq M$.

□

Limits and Comparisons (4 points)

Exercise 7.

Consider $f(x) = \dfrac{(1+x^5)\ln(1+\frac{1}{x^2})}{2x^2 - 1}$.

1. Determine the domain of definition of f.
2. Provide an equivalent of f at $\pm\infty$.

3. (*) Show that f admits an oblique asymptote at $\pm\infty$ and identify its local position.
4. Does f admit any vertical asymptote? If so provide them and the associated limits.
5. Graph f.

Solution. 1. $D_f = \mathbb{R} \setminus \{-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\}$.

2. $\ln(1+\frac{1}{x^2}) \underset{\infty}{\sim} \frac{1}{x^2}$, $1+x^5 \underset{\infty}{\sim} x^5$, $2x^2-1 \underset{\infty}{\sim} 2x^2$ so $f(x) \underset{\infty}{\sim} \frac{x^3}{2x^2} = \frac{x}{2} \xrightarrow[x\to\pm\infty]{} \pm\infty$.

3. Let's do some asymptotic expansions of $\frac{f(x)}{x}$.

$$\begin{aligned}
\frac{f(x)}{x} = \frac{(1+x^5)\ln(1+\frac{1}{x^2})}{2x^3-x} &\underset{\infty}{=} \frac{(1+x^5)\left(\frac{1}{x^2}-\frac{1}{2x^4}+\frac{1}{3x^6}+o(\frac{1}{x^6})\right)}{2x^3(1-\frac{1}{2x^2})} \\
&\underset{\infty}{=} \frac{1+x^5}{2x^3}\left(\frac{1}{x^2}-\frac{1}{2x^4}+\frac{1}{3x^6}+o(\frac{1}{x^6})\right)(1+\frac{1}{2x^2}+\frac{1}{4x^4}+\frac{1}{8x^6}+o(\frac{1}{x^6})) \\
&\underset{\infty}{=} (\frac{1}{2x^3}+\frac{x^2}{2})\left(\frac{1}{x^2}+\frac{1}{3x^6}++o(\frac{1}{x^6})\right) \\
&\underset{\infty}{=} \frac{1}{2}+\frac{1}{6x^4}+o(\frac{1}{x^4})
\end{aligned}$$

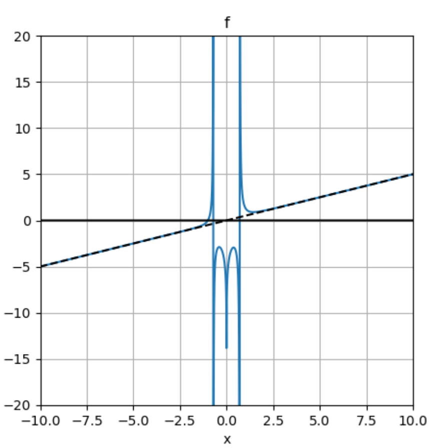
so we find

$$f(x) \underset{\infty}{=} \frac{x}{2}+0+\frac{1}{6x^3}+o(\frac{1}{x^3})$$

so f has the line $y=\frac{x}{2}$ as oblique asymptote at $\pm\infty$. The next term $\frac{1}{6x^3}$ gives us that the graph of f is below the asymptote at $-\infty$ and above at $+\infty$.

4. f admits three vertical asymptotes, at $-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}$. We find

$$\lim_{x\to 0^+} f(x) = -\infty, \qquad \lim_{x\to -\frac{1}{\sqrt{2}}^\pm} f(x) = \mp\infty, \qquad \lim_{x\to \frac{1}{\sqrt{2}}^\pm} f(x) = \pm\infty$$

5. 

□

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