Exam n° 3 – 1 hour 30 minutes

— No documents, no calculators, no cell phones or electronic devices allowed.
— Take a deep breath before starting (everything is going to be ok!) and read entirely the exam before starting.[0]
— All exercises are independent, you can do them in the order that you'd like.
— Please start an exercise at the top of a page (for readability).
— Number single pages, or simply the booklets (*copies doubles*) if multiple : for example 1/3, 2/3, 3/3
— All your answers must be fully (but concisely) justified, unless noted otherwise.
— Redaction and presentation matter! For instance, write full sentences and make sure your 'x' and 'n' can be distinguished.
— Questions marked (*) are considered more challenging.
— **Respecting all of the above is part of the exam grade (0.5 points).** Provided rubric is indicative (changes may occur).

Fundamental exercises (11,5 points)

You are expected to provide some steps for those exercises. Little partial credit will be given for just writing the answer.

Exercise 1. Does it make sense or not? (2 points) Justify briefly if the following statements use a valid reasoning or not, or if they are True or False.
If it's not a valid reasoning, simply indicate where is the problem (or what is missing).
If it's False, provide the correct answer (if applicable).

1. *Valid reasoning?* Consider $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x+y) = f(x) + f(y)$, $\forall (x,y) \in \mathbb{R}^2$. Then f is a linear application.
2. *Valid reasoning?* Consider $f \in \mathcal{L}(\mathbb{R}^n)$, such that $\forall x \in \mathbb{R}^n$ $f^4(x) = f(x)$. Since $f^4 = f^2 \circ f^2 = f$, then f is a projection.
3. *True or False?* Using equivalents we find that $\sin(x) - x \underset{0}{\sim} 0$.
4. *True or False?* If $F = \text{Span}((1,0))$ and $G = \text{Span}((-1,1))$ then $F \oplus G = \mathbb{R}^2$.

Exercise 2. (1.5 points) Solve $\cos(2x) = -1 + \sin(2x)$ over $\mathbb{R}$.

Exercise 3. (3 points) Provide an equivalent of the following functions (justify your steps) and deduce their limit.

$$f(x) = \frac{x^2 \ln(\frac{1+x}{x}) + \ln(x^3)}{x^3(1-2x)} \quad \text{at } +\infty$$

$$g(x) = \frac{\ln(1-\tan(x))}{\sin(3x)} \quad \text{at } 0$$

Exercise 4. (2.5 points) Consider $s = (1,0,1)$, $t = (1,1,1)$, $u = (0,-2,0)$.

1. Compute the rank of the family (s,t,u).

2. Is this family a basis of $\mathbb{R}^3$? Justify your answer. In case it is not, complete the linearly independent subfamily to form a basis of $\mathbb{R}^3$. We will call this family B.
3. Consider $v = (x, y, z)$ expressed in the canonical basis of $\mathbb{R}^3$. Express $[v]_B$, that is v in the basis B.

Exercise 5. (2.5 points) We want to determine the regularity of the function defined over $\mathbb{R}$ by

$$f(x) = \begin{cases} \sin(x^3) - x^3 & x \leq 0 \\ x^6 \ln(1+x) & x > 0 \end{cases}$$

To that aim we define $f_1(x) = \sin(x^3) - x^3$ for $x \leq 0$, and $f_2(x) = x^6 \ln(1+x)$ for $x \geq 0$.

1. What is the regularity of f_1, f_2 ?
2. Based on previous question provide their associated Taylor series expansions at $x = 0$, with at least one non-zeros term and both of the same order.
3. How can we deduce the TSEs of their respective derivatives at 0 ? From this find n, m such that $f_1^{(n)}(0) \neq 0$, $f_2^{(m)}(0) \neq 0$.
4. Deduce the class of the function f.

Linear applications (4 points)

Exercise 6.
Consider the linear application $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that $f(x, y, z) = (x - 2y + 3z, -6z - 2x + 4y, -2z - x)$.

1. Write the associated matrix M relative to the canonical basis of $\mathbb{R}^3$.
2. Without computing neither the kernel or the image, determine the dimension of one of them (state explicitly which one you pick), and **deduce** the dimension of the other. Justify your answer.
3. Can f be injective ? Surjective ? Bijective ? Justify each answer.
4. Provide a basis K of $\ker(f)$, and a basis I of $\text{Im}(f)$. We expect proper justification here.
5. Show that $\ker(f) \oplus \text{Im}(f) = \mathbb{R}^3$.
6. (*) Can we conclude that f is a projection ? Justify your answer.

Limits and Comparisons (4 points)

Exercise 7.
Consider $f(x) = \dfrac{(1 + x^5) \ln(1 + \frac{1}{x^2})}{2x^2 - 1}$.

1. Determine the domain of definition of f.
2. Provide an equivalent of f at $\pm\infty$.
3. (*) Show that f admits an oblique asymptote at $\pm\infty$ and identify its local position.
4. Does f admit any vertical asymptote ? If so provide them and the associated limits.
5. Graph f.

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0. Draw a skull next to your name on the first page once this is done.