Exam nº 1 – SOLUTIONS

- No documents, no calculators, no cell phones or electronic devices allowed.
- Take a deep breath before starting (everything is going to be ok!) and read entirely the exam before starting. [1]
- All exercises are independent, you can do them in the order that you'd like.
- Please start an exercise at the top of a page (for readability).
- Number single pages, or simply the booklets (*copies doubles*) if multiple : for example 1/3, 2/3, 3/3
- All your answers must be fully (but concisely) justified, unless noted otherwise.
- Redaction and presentation matter! For instance, write full sentences and make sure your 'x' and 'n' can be distinguished.
- Questions marked (*) are considered more challenging.
- **Respecting all of the above is part of the exam grade (0.5 points).** Provided rubric is indicative (changes may occur).

Fundamental exercises (10.5 points)

Fundamental exercises test the expected learning outcomes you should have gained.

You are expected to provide some steps for those exercises. Little partial credit will be given for just writing the answer.

Exercise 1. True or False (4.5 points) Justify briefly why the statement is True or False. If the statement is false, provide the correct answer.

1. The negation of $(\exists\varepsilon > 0, \forall x \geq x_0, |x-x_0| \leq \varepsilon \Rightarrow |f(x)-\ell| \leq \varepsilon)$ is $(\exists\varepsilon \leq 0, \exists x \geq x_0, |f(x)-\ell| > \varepsilon \Rightarrow |x-x_0| \leq \varepsilon)$.
2. The contrapositive of $(x^2+1=2 \Rightarrow x=1)$ is $(x=1 \Rightarrow x^2=1)$.
3. Let $A = \{(x,y) \in \mathbb{R}^2 | x+y=2\}$, $B = \{(x,y) \in \mathbb{R}^2 | x=\alpha,\ y=2-\alpha,\ \alpha \in \mathbb{R}\}$. Then $A=B$.
4. Consider the sets A, B, C with $A, B, C \subset E$. Then $(A \cup B) \cap ((E \setminus B) \cup C) = A \cup B \cup C$.
5. Let $n \in \mathbb{N}$, $n > 1$, and $x \in (0, +\infty)$. Then $\left(\frac{1}{x}+1\right)^n = 1 + \frac{1}{x^n} + \sum_{k=1}^{n-1} \binom{n}{k} x^{k-n}$.
6. $f^{-1} : F \to E$ is injective means $\forall x, x' \in F$, $x = x' \Rightarrow f^{-1}(x) = f^{-1}(x')$.
7. The interval $[-2, 3)$ admits a maximum equal to 3.

Solution.

1. False : the negation of $(\exists\varepsilon > 0, \forall x \geq x_0, |x-x_0| \leq \varepsilon \Rightarrow |f(x)-\ell| \leq \varepsilon)$ is $(\forall\varepsilon > 0, \exists x \geq x_0, |f(x)-\ell| > \varepsilon$ and $|x-x_0| \leq \varepsilon)$.
2. False : the contrapositive of $(x^2+1=2 \Rightarrow x=1)$ is $(x \neq 1 \Rightarrow x^2 \neq 1)$.
3. True : Let $(x,y) \in A$. Then $x+y=2 \Leftrightarrow x=x$ and $y=2-x$ and $(x, 2-y) \in B$. Let $(x,y) \in B$. Then $x+y = \alpha+2-\alpha = 2$ so $(x, y \in A)$. Therefore $A=B$.
4. False : by distributivity $(A \cup B) \cap ((E \setminus B) \cup C) = (A \cup \underbrace{B \cap E \setminus B}_{\emptyset}) \cup (A \cup B \cap C) = A \cup B \cap C$.

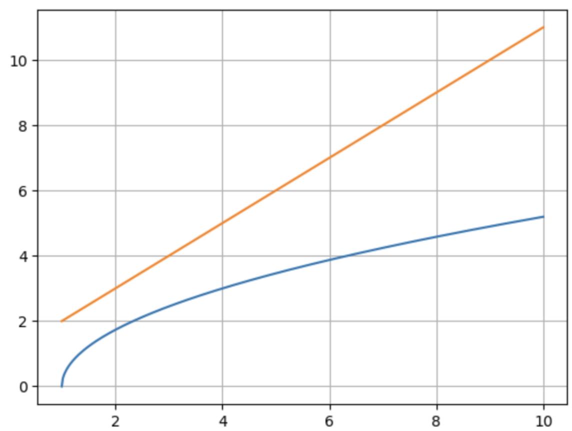
1. Draw a ghost next to your name on the first page once this is done.

5. True : this is Newton's formula, $\left(\frac{1}{x}+1\right)^n = \sum_{k=0}^{n} \binom{n}{k} x^{k-n} = 1 + \frac{1}{x^n} + \sum_{k=1}^{n-1} \binom{n}{k} x^{k-n}$.
6. False : the definition of f^{-1} being injective is $\forall x, x' \in F,\ f^{-1}(x) = f^{-1}(x') \Rightarrow x = x'$.
7. False : the interval $[-2, 3)$ has no maximum (but a supremum equal to 3).

□

Exercise 2. (2 points)
Solve the inequation $\sqrt{3x-3} \geq |x+1|$ in $\mathbb{R}$.

Solution.
— The function $x \mapsto \sqrt{3x-3}$ is only defined when $3x - 3 \geq 0 \Leftrightarrow x \geq 1$.
— The functions $x \mapsto |x+1|$ is defined over $\mathbb{R}$ and $|x+1| = x+1$ for $x \geq -1$ and $-x-1$ for $x < -1$.
— Assume $x \geq 1$. Then the functions are defined and positive (and there is no need to do a case disjunction about the absolute value) : $\sqrt{3x-3} \geq |x+1| \Leftrightarrow \sqrt{3x-3} \geq x+1 \Rightarrow 3x-3 \geq x^2+2x+1 \Leftrightarrow x^2 - x + 4 \leq 0$.
— We study the sign of the polynomial $x^2 - x + 4$. $\Delta = 1 - 8 \leq 0$ so it has no roots, therefore $x^2 - x + 4 \geq 0$ for all $x \leq 1$.
— We conclude there are no solutions to this inequation. A graphical representation of those 2 functions shows indeed it is never the case.



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Exercise 3. (2 points)
1. Solve directly (no change of formula) the equation $\cos(3x) = \cos(x)$ in $\mathbb{R}$.
2. Factorize $\cos(3x)$ by $\cos(x)$ then solve the equation $\cos(3x) = \cos(x)$. What do you conclude?

Solution.
— $\cos(3x) = \cos(x) \Leftrightarrow \begin{cases} 3x = x + 2\pi k \\ 3x = -x + 2\pi k \end{cases}, k \in \mathbb{Z} \Leftrightarrow \begin{cases} x = \pi k \\ x = \frac{\pi}{2} k \end{cases}, k \in \mathbb{Z}$ so the set of solutions is $S = \frac{\pi}{2}\mathbb{Z}$ (indeed $\pi\mathbb{Z} \subset \frac{\pi}{2}\mathbb{Z}$).
— $\cos(3x) = \cos(2x)\cos(x) - \sin(2x)\sin(x) = \cos(x)(\cos(2x) - 2\sin^2(x)) = \cos(x)(1 - 4\sin^2(x))$.
— Then $\cos(3x) = \cos(x) \Leftrightarrow \cos(x)(1 - 4\sin^2(x)) = \cos(x) \Leftrightarrow \cos(x)\sin^2(x) = 0 \Leftrightarrow x = \frac{\pi}{2} + \pi k$ or $x = \pi k$, $k \in \mathbb{Z}$.
— We conclude the same set of solutions.

□

Exercise 4. (2 points)
Let $n \in \mathbb{N}^*$, $U_n = \sum_{k=1}^{n} (2n + 2k + 1)$, $V_n = \sum_{k=1}^{n} \left((n+1+k)^2 - (n+k)^2\right)$, and $W_n = \sum_{k=1}^{n} (n+k)^2$.

1. Compute U_n.
2. Show that $U_n = V_n$.
3. Show that $W_{n+1} - W_n = 7n^2 + 10n + 4$.

Solution. — $U_n = \sum_{k=1}^{n}(2n+2k+1) = (2n+1)\sum_{k=1}^{n} 1 + 2\sum_{k=1}^{n} k = 2n^2 + n + n(n+1) = 3n^2 + 2n$.

— $V_n = \sum_{k=1}^{n}\left((n+1+k)^2 - (n+k)^2\right) = \sum_{k=1}^{n}(n+1+k+n+k)(n+1+k-n-k) = \sum_{k=1}^{n}(2n+$ $+2k+1) = U_n$. *Doing by telescopic sums is also a valid approach.*

— $W_{n+1} - W_n = \sum_{k=1}^{n+1}(n+1+k)^2 - \sum_{k=1}^{n}(n+k)^2 = (2n+2)^2 + V_n = 4(n^2+2n+1) + 3n^2 + 2n =$ $7n^2 + 10n + 4$.

□

Functions (5.5 points)

Exercise 5. Let f the function defined by $f(x) = \ln\left(\frac{x+2}{x-2}\right)$.

1. Find D_f the domain of definition of the function. Provide steps.
2. (*) Show that f is odd over D_f.
3. We now restrict the domain of f to $D_f \cap \mathbb{R}^+$. We denote $\tilde{f}$ this function. Consider the functions $f_1 : x \mapsto \ln(x)$, $f_2 : x \mapsto 1 + 4x$, $f_3 : x \mapsto \frac{1}{x-2}$.
   (a) Does $\tilde{f}$ admit horizontal and/or vertical asymptotes ? If yes, provide then.
   (b) Provide domains and codomains of f_1, f_2, f_3 so that $\tilde{f} = f_1 \circ f_2 \circ f_3$.
   (c) Study monotonicity of f_1, f_2, f_3 then conclude about the monotonicity of $\tilde{f}$.
4. Choose a codomain so that $\tilde{f}$ is surjective.
5. Is the function $\tilde{f}$ bijective ? Justify your answer.
6. (*) Sketch the function f over $\mathbb{R}$.

Solution. — The function is defined when $\frac{x+2}{x-2} > 0 \Rightarrow x \in D_f := (-\infty, -2) \cup (2, +\infty)$. *Do not split the log otherwise you miss the point.*

— D_f is symmetric ($\forall x \in D_f,\ -x \in D_f$), and $\forall x \in D_f,\ f(-x) = \ln\left(\frac{-x+2}{-x-2}\right) = \ln\left(\frac{x-2}{x+2}\right) =$ $-\ln\left(\frac{x+2}{x-2}\right) = -f(x)$. So f is odd over D_f.

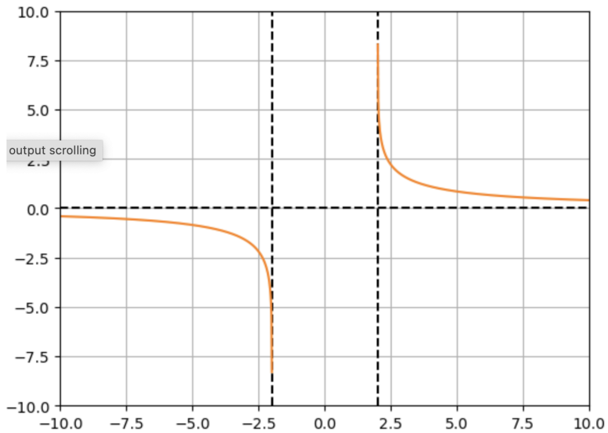
— $\tilde{f}$ admits a vertical asymptote at 2^+ ($\lim_{x\to 2^+} \ln\left(\frac{x+2}{x-2}\right) = \frac{4}{0^+} = +\infty$), and a horizontal asymptote at $+\infty$ ($\lim_{x\to+\infty} \ln\left(\frac{x+2}{x-2}\right) = 0$).

— We choose $f_3 : (2, +\infty) \to \mathbb{R}_*^+$, $f_2 : \mathbb{R}_*^+ \to (1, +\infty)$, and $f_1 : (1, +\infty) \to \mathbb{R}_*^+$.

— f_3 is decreasing over $(2, +\infty)$ and f_2 is increasing over $\mathbb{R}_*^+$ so $f_2 \circ f_3$ is decreasing over $(2, +\infty)$. f_1 is increasing over $(1, +\infty)$ so $f_1 \circ f_2 \circ f_3$ is decreasing over $(2, +\infty)$.

— We deduce $\tilde{f} : (2 + \infty) \to \mathbb{R}_*^+$ is surjective.

— $\tilde{f}$ is even strictly decreasing over $(2, +\infty)$ so it is injective, and surjective per choice of codomain. So it is bijective.

— 

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Logic (3.5 points)

Exercise 6.

Let $f : \mathbb{R} \to \mathbb{R}$ and consider the following statements :

$$A \equiv \forall x \in \mathbb{R},\ f(x+1) = f(x) + 1, \qquad B \equiv \exists a \in \mathbb{R},\ \forall n \in \mathbb{N},\ f(n) = a + n$$

1. Write the negation of B.
2. Provide the negation and contrapositive of $A \Rightarrow B$.
3. Assume A is true. Show by induction that $\forall n \in \mathbb{N},\ f(n) = f(0)+n$. We expect proper redaction.
4. Is $A \Rightarrow B$ true ? Justify your answer.

Solution. — $\neg B \equiv \forall a \in \mathbb{R}, \exists n \in \mathbb{N},\ f(n) \neq a + n$.

— $\neg(A \Rightarrow B) \Leftrightarrow A \wedge \neg B \equiv (\forall x \in \mathbb{R},\ f(x+1) = f(x) + 1) \wedge (\forall a \in \mathbb{R}, \exists n \in \mathbb{N},\ f(n) \neq a + n)$. The contrapositive of $A \Rightarrow B$ is $\neg B \Rightarrow \neg A \equiv (\forall a \in \mathbb{R}, \exists n \in \mathbb{N},\ f(n) \neq a + n) \Rightarrow (\exists x \in \mathbb{R},\ f(x+1) \neq f(x) + 1)$. *Partial credit for just giving the definition.*

— We want to show that $\forall n \in \mathbb{N},\ P(n) : f(n) = f(0) + n$ is true.
  - — For $n = 0$, we have $f(0) = f(0) + 0$ so $P(0)$ true.
  - — Assume $P(n)$ true. We want to show $P(n+1)$ true, that is $f(n+1) = f(0) + n + 1$. Because A is true for $x \in \mathbb{R}$, in particular A is true for $n \in \mathbb{N}$. We deduce that $f(n+1) = f(n) + 1$, and by $P(n)$ we have $f(n) = f(0) + n$, so $f(n+1) = f(0) + n + 1$.
  - — We conclude that $\forall n \in \mathbb{N},\ f(n) = f(0) + n$.

— Assume A true. Then in particular $\forall n \in \mathbb{N},\ f(n+1) = f(n) + 1$, which implies per previous question that $f(n) = f(0) + n$. We conclude that $a = f(0)$ works so B true.

□
