Exam n° 1 – 1 hour 30 minutes

- No documents, no calculators, no cell phones or electronic devices allowed.
- Take a deep breath before starting (everything is going to be ok!) and **read entirely the exam before starting.** [1]
- All exercises are independent, you can do them in the order that you'd like.
- Please start an exercise at the top of a page (for readability).
- Number single pages, or simply the booklets (*copies doubles*) if multiple : for example 1/3, 2/3, 3/3
- All your answers must be fully (but concisely) justified, unless noted otherwise.
- Redaction and presentation matter! For instance, write full sentences and make sure your 'x' and 'n' can be distinguished.
- Questions marked (*) are considered more challenging.
- **Respecting all of the above is part of the exam grade (0.5 points).** Provided rubric is indicative (changes may occur).

Fundamental exercises (10.5 points)

Fundamental exercises test the expected learning outcomes you should have gained.

You are expected to provide some steps for those exercises. Little partial credit will be given for just writing the answer.

Exercise 1. True or False (4.5 points) Justify briefly why the statement is True or False. If the statement is false, provide the correct answer.

1. The negation of $(\exists \varepsilon > 0, \forall x \geq x_0, |x - x_0| \leq \varepsilon \Rightarrow |f(x) - \ell| \leq \varepsilon)$ is $(\exists \varepsilon \leq 0, \exists x \geq x_0, |f(x) - \ell| > \varepsilon \Rightarrow |x - x_0| \leq \varepsilon)$.
2. The contrapositive of $(x^2 + 1 = 2 \Rightarrow x = 1)$ is $(x = 1 \Rightarrow x^2 = 1)$.
3. Let $A = \{(x, y) \in \mathbb{R}^2 | x + y = 2\}$, $B = \{(x, y) \in \mathbb{R}^2 | x = \alpha,\ y = 2 - \alpha,\ \alpha \in \mathbb{R}\}$. Then $A = B$.
4. Consider the sets A, B, C with $A, B, C \subset E$. Then $(A \cup B) \cap ((E \setminus B) \cup C) = A \cup B \cup C$.
5. Let $n \in \mathbb{N}$, $n > 1$, and $x \in (0, +\infty)$. Then $\left(\dfrac{1}{x} + 1\right)^n = 1 + \dfrac{1}{x^n} + \displaystyle\sum_{k=1}^{n-1} \binom{n}{k} x^{k-n}$.
6. $f^{-1} : F \to E$ is injective means $\forall x, x' \in F,\ x = x' \Rightarrow f^{-1}(x) = f^{-1}(x')$.
7. The interval $[-2, 3)$ admits a maximum equal to 3.

1. Draw a ghost next to your name on the first page once this is done.

Exercise 2. (2 points)
Solve the inequation $\sqrt{3x-3} \geq |x+1|$ in $\mathbb{R}$.
Exercise 3. (2 points)

1. Solve directly (no change of formula) the equation $\cos(3x) = \cos(x)$ in $\mathbb{R}$.
2. Factorize $\cos(3x)$ by $\cos(x)$ then solve the equation $\cos(3x) = \cos(x)$. What do you conclude?

Exercise 4. (2 points)
Let $n \in \mathbb{N}^*$, $U_n = \sum_{k=1}^{n}(2n+2k+1)$, $V_n = \sum_{k=1}^{n}\left((n+1+k)^2 - (n+k)^2\right)$, and $W_n = \sum_{k=1}^{n}(n+k)^2$.

1. Compute U_n.
2. Show that $U_n = V_n$.
3. Show that $W_{n+1} - W_n = 7n^2 + 10n + 4$.

Functions (5.5 points)

Exercise 5. Let f the function defined by $f(x) = \ln\left(\frac{x+2}{x-2}\right)$.

1. Find D_f the domain of definition of the function. Provide steps.
2. (*) Show that f is odd over D_f.
3. We now restrict the domain of f to $D_f \cap \mathbb{R}^+$. We denote $\tilde{f}$ this function. Consider the functions $f_1 : x \mapsto \ln(x)$, $f_2 : x \mapsto 1 + 4x$, $f_3 : x \mapsto \frac{1}{x-2}$.
   (a) Does $\tilde{f}$ admit horizontal and/or vertical asymptotes? If yes, provide then.
   (b) Provide domains and codomains of f_1, f_2, f_3 so that $\tilde{f} = f_1 \circ f_2 \circ f_3$.
   (c) **Without computing the derivatives**, study monotonicity of f_1, f_2, f_3 then conclude about the monotonicity of $\tilde{f}$.
4. Choose a codomain so that $\tilde{f}$ is surjective.
5. Is the function $\tilde{f}$ bijective? Justify your answer.
6. (*) Sketch the function f over $\mathbb{R}$.

Logic (3.5 points)

Exercise 6.
Let $f : \mathbb{R} \to \mathbb{R}$ and consider the following statements :

$$A \equiv \forall x \in \mathbb{R},\ f(x+1) = f(x) + 1, \qquad B \equiv \exists a \in \mathbb{R}, \forall n \in \mathbb{N},\ f(n) = a + n$$

1. Write the negation of B.
2. Provide the negation and contrapositive of $A \Rightarrow B$.
3. Assume A is true. Show by induction that $\forall n \in \mathbb{N},\ f(n) = f(0) + n$. We expect proper redaction.
4. Is $A \Rightarrow B$ true? Justify your answer.
