Test #2, December 1st, 2025 - Duration 1h30

Warnings and advice

- All documents, calculators or electronic devices, means of communication, dictionnaries, are prohibited.
- The grading scale is given as an indication.
- Presentation, quality of writing, clarity and precision of reasoning are taken into account in the grading.

Exercise 1 ***Limits (2 pts)***

Determine the limits, if any, of the sequences below :

$$u_n = \sqrt{n^2 + \sqrt{n}} - n \qquad\qquad v_n = \left(1 - \frac{2}{n}\right)^n$$

Exercise 2 ***Determinants (4 pts)***

The two questions in this exercise are independent.

1) Let $A \in \mathcal{M}_n(\mathbb{R})$ be such that $A^2 = 2A$. What are the possible values for the determinant of A?
2) a) Let $a \in \mathbb{R}$. Compute the determinant of the matrix $M(a)$ below :

$$M(a) = \begin{pmatrix} 2-a & -1 & -1 \\ -1 & 2-a & 1 \\ -1 & 1 & 2-a \end{pmatrix}.$$

   b) For which values of a is the matrix $M(a)$ invertible?

   c) What are the eigenvalues of the matrix $A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}$?

   d) Without any additional computation, what can be said about the dimension of the associated eigenspaces?

Exercise 3 ***Adjacent Sequences (3.5 pts)***

Let a and b be two real numbers such that $a > b > 0$.
We define the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ by

$$\begin{cases} x_0 = a \\ \forall n \in \mathbb{N},\ x_{n+1} = \dfrac{a\,x_n + b\,y_n}{a+b} \end{cases} \qquad\qquad \begin{cases} y_0 = b \\ \forall n \in \mathbb{N},\ y_{n+1} = \dfrac{a\,y_n + b\,x_n}{a+b}. \end{cases}$$

1) Let $(w_n)_{n\in\mathbb{N}}$ be the sequence defined by : $\forall n \in \mathbb{N},\ w_n = y_n - x_n$.
   Show that $(w_n)_{n\in\mathbb{N}}$ is a geometric sequence.
2) Show that the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are adjacent.
   What can be deduced from this?
3) Study the sequence $(x_n + y_n)_{n\in\mathbb{N}}$.
   Deduce the limit of $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$.

EXERCISE 4 ***Iterative Relations (10.5 pts)***

Let f be the function defined by : $\forall x \in \mathbb{R},\ f(x) = \dfrac{1}{2}\, x\,(1-x)$ and $(u_n)_{n\in\mathbb{N}}$ the sequence defined by

$$\begin{cases} u_0 = a \in \mathbb{R} \\ \forall n \in \mathbb{N},\ u_{n+1} = f(u_n). \end{cases}$$

Part One

1) a) Study the sign of $f(x) - x$.
   b) Study the variations of f then sketch the curve of f.

2) We assume in this question that $a \in \left[0, \dfrac{1}{2}\right]$.

   a) Show that for all $n \in \mathbb{N}, u_n \in \left[0, \dfrac{1}{2}\right]$.
   b) Study the variations of $(u_n)_{n\in\mathbb{N}}$.
   c) Study the convergence of $(u_n)_{n\in\mathbb{N}}$ and give its limit, if any.
   d) What happens if $a \in \left]\dfrac{1}{2}, 1\right[$?

3) Determine the variations and the limit (if any) of $(u_n)_{n\in\mathbb{N}}$ when $a < -1$.

Part Two

We now consider the case where $a \in]0,1[$. We thus have $\lim\limits_{n\to+\infty} u_n = 0$ and for all $n \in \mathbb{N}, u_n > 0$.
We define the sequence $(w_n)_{n\in\mathbb{N}}$ by : $\forall n \in \mathbb{N},\ w_n = 2^n u_n$.

1) Show that for all $n \in \mathbb{N}$, $w_{n+1} - w_n = -\dfrac{w_n^2}{2^n}$.
2) Deduce that $(w_n)_{n\in\mathbb{N}}$ converges to a limit $\ell \in [0,1[$.
3) Let $n \in \mathbb{N}^*$.

   a) Calculate the sum $\displaystyle\sum_{k=1}^{n-1} \frac{1}{2^k}$.
   b) Deduce that $\displaystyle\sum_{k=1}^{n-1} \frac{w_k^2}{2^k} \leqslant w_1^2$.
   c) Deduce that $w_1 - w_1^2 \leqslant w_n \leqslant w_1$.

4) Deduce that $\ell > 0$, and that $u_n \underset{n\to+\infty}{\sim} \dfrac{\ell}{2^n}$.