INSA LYON
FIMI – SCAN2

INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

TEST #1, OCTOBER 20, 2025 - DURATION 1H30

Warnings and advice

- All documents, calculators or electronic devices, means of communication, dictionnaries, are prohibited.
- The grading scale is given as an indication.
- Presentation, quality of writing, clarity and precision of reasoning are taken into account in the grading.

EXERCISE 1 *(4 pts)*

Study the nature of the following integrals :

1) $\displaystyle\int_1^{+\infty} \frac{x\sqrt{x}}{1+x^3} \sin\left(x^2\right) \mathrm{d}x$ 2) $\displaystyle\int_0^{+\infty} \frac{1+x^{2025}}{\sqrt{x}\ \mathrm{e}^x} \mathrm{d}x$ 3) $\displaystyle\int_0^1 \frac{x^2}{\ln(x)} \mathrm{d}x$

EXERCISE 2 *(7 pts)*

1) Compute $c_n = \displaystyle\int_1^{+\infty} \frac{\ln t}{t^n} \mathrm{d}t$ for all integers $n \geqslant 2$.
2) For which values of $n \in \mathbb{N}$ does the integral

$$a_n = \int_1^{+\infty} \frac{\ln t}{t^n(t+1)} \mathrm{d}t$$

converge ?
3) Show that the sequence $(a_n)_{n\geqslant 1}$ is strictly decreasing and convergent.
4) By comparison with the sequence (c_n), determine the limit of (a_n).
5) Compute the sum $a_n + a_{n+1}$ and deduce that for all $n \geqslant 2$,

$$\frac{1}{2n^2} \leqslant a_n \leqslant \frac{1}{2(n-1)^2}.$$

6) Determine an equivalent of the sequence (a_n).

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EXERCISE 3 *(9 pts)*

Preliminary question. Let $t \in [1, +\infty)$ be fixed and define for all $x \in [0,1]$, $\varphi(x) = t^{x-1} + t^{1-x}$.
Show that φ is decreasing on $[0,1]$.

For $x \in \mathbb{R}$ such that the integral converges, set $f(x) = \displaystyle\int_0^{+\infty} \frac{t^{x-1}}{1+t^2}\,\mathrm{d}t$.

1) Determine the domain D of the function f, i.e., the set of values of x for which the integral converges.

2) a) By performing the substitution $u = \dfrac{1}{t}$, show that for all $x \in D$, $\quad f(2-x) = f(x)$.

   b) What symmetry property can be deduced for the graph of f?

3) a) Using a substitution, show that for all $x \in D$

$$\int_0^1 \frac{t^{x-1}}{1+t^2}\,\mathrm{d}t = \int_1^{+\infty} \frac{t^{1-x}}{1+t^2}\,\mathrm{d}t$$

   b) Deduce that for all $x \in D$,

$$f(x) = \int_1^{+\infty} \frac{t^{x-1} + t^{1-x}}{1+t^2}\,\mathrm{d}t.$$

   c) Using the preliminary question, show that f is decreasing on $(0,1]$.

4) Deduce from the previous questions that f admits a minimum on D and compute this minimum.

5) a) Justify that for any continuous and positive function g on $[0,+\infty)$ such that $\displaystyle\int_0^{+\infty} g(t)\,\mathrm{d}t$ converges, we have :

$$\int_0^{+\infty} g(t)\,\mathrm{d}t \geqslant \int_0^1 g(t)\,\mathrm{d}t.$$

   b) Show that for all $x \in D$, $f(x) \geqslant \dfrac{1}{2x}$.

   c) Determine the limits of f at the boundaries of D.