Written physics Test 1

The two exercises are completely independent and the questions within the exercises are mostly independent. ***Unjustified results will not be taken into account.*** *Care should be taken to ensure that your manuscript is presented clearly and legibly. It is recalled that the use of any internet-connected object is strictly forbidden*

A formula sheet, useful for both exercises, is provided at the end of the text.

Part 1 Vector fields

Part 1.1 Flow through a sphere

Consider the vector field $\vec{A}$ defined in a Cartesian coordinate system $(O, \vec{u}_x, \vec{u}_y, \vec{u}_z)$ by :

$$\vec{A} = x^3\,\vec{u}_x + y^3\,\vec{u}_y + z^3\,\vec{u}_z$$

Question 1 : *Calculate the divergence of this vector field.*

Question 2 : *Calculate the flux of this vector field through a sphere of radius R and center O, origin of the coordinate system.*

Part 1.2 Vector field in polar coordinates

Consider the vector field $\vec{E}$ defined **in polar coordinates** $(\vec{u}_r, \vec{u}_\theta)$ by the following components :

$$E_r = \frac{2k\cos(\theta)}{r^3} \quad E_\theta = \frac{k\sin(\theta)}{r^3}$$

where k is a positive constant.

Question 3 : *Does this vector field derive from a scalar potential V ? If so, calculate this potential. Care should be taken to detail all the calculations.*

Question 4 : *Determine the equation of the field lines of $\vec{E}$ and draw these field lines schematically (given : $\frac{d}{d\theta}\left(\ln(\sin\theta)\right) = \frac{\cos\theta}{\sin\theta}$)*

Part 2 Electrostatics

In this part, suppose you're helping out a friend who is in trouble with physics. Your friend has tried to solve the three following exercises but has run into problems with the solutions.
What you have to do :

1. provide your own full solution for each exercise, and then,
2. identify what was the problem in your friend's solution

You will take particular care to give a detailed and well-argued response.

Exercise $\#1$
Consider a sphere of radius R containing a uniform volume charge density ρ in a space with permittivity ϵ : determine the electric field $\vec{E}$ created throughout space by this charge distribution.

Answer to be verified :
Let's use a Cartesian coordinate system with the centre O of the sphere as the origin and spherical coordinates (r, θ, ϕ). At a point M distinct from O, any plane containing the line OM is a plane of symmetry of the charge distribution, and therefore contains $\vec{E}(M)$, which is collinear with $\vec{OM}$ and therefore with $\vec{u}_r$. Furthermore, any rotation by an angle θ or ϕ leaves the charge distribution unchanged, so $\vec{E}(M) = E_r(r)\, \vec{u}_r$.

If $r < R$, M is in the ball, so $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon}$.

Thus $\dfrac{\mathrm{d}E_r}{\mathrm{d}r} = \dfrac{\rho}{\epsilon}$ and $E_r = \dfrac{\rho r}{\epsilon} + c$, with c being a constant.
If $r > R$, there is no charge, so $\vec{\nabla} \cdot \vec{E} = 0$ and thus $E_r = cte$. By continuity of $\vec{E}$ at $r = R$, $E_r(r > R) = \dfrac{\rho R}{\epsilon} + c$.
However, the field $\vec{E}$ must be zero far away from the charged source, so $\frac{\rho R}{\epsilon} + c = 0$, which determines the constant.

Exercise $\#2$
Consider an infinitely long cylinder of radius R carrying a uniform surface charge density σ in a space of permittivity ϵ, in a cylindrical coordinate system with axis Oz coinciding with the axis of the cylinder. Assume that the field $\vec{E}$ is radial anywhere throughout space. It is given outside the cylinder $(r > R)$: $\vec{E} = \dfrac{\sigma R}{\epsilon r}\vec{u}_r$ and we want to determine it inside the cylinder.

Answer to be verified :
The surface $r = R$ is a charged surface, its normal is $\vec{u}_r$, so the electric field $\vec{E}$ is normal to the charged surface and undergoes a discontinuity of $\dfrac{\sigma}{\epsilon}\, \vec{u}_r$ at the interface. Thus, the electric field inside the cylinder $(r < R)$ is $\vec{E} = \left(\dfrac{\sigma R}{\epsilon r} - \dfrac{\sigma}{\epsilon}\right) \vec{u}_r$.

Exercise #3
Between the planes $z = -h$ and $z = 0$, a uniform volume charge density $-\rho$ is placed; between the planes $z = 0$ and $z = h$, a uniform volume charge density $+\rho$ is placed. Both volume charge distributions are infinite along x and y. Calculate the electric field $\vec{E}$ created by this global charge distribution throughout space with permittivity ϵ.

Hint : you can refer to an exercise done in the tutorials and use the superposition principle : the global electric field is the vector sum of the electric fields created by the two volume charge distributions considered separately. **Perform a diagram of the various electric fields as a function of z.**

Answer to be verified :
Let's apply the principle of superposition and first calculate the field $\vec{E}_1$ created by the positive volume charge distribution located between $z = 0$ and $z = h$. The charges are invariant under translation along Ox or Oy, so $\vec{E}_1$ depends only on z. At any point M, the planes (Mxz) and (Myz) are planes of symmetry of the charges, so $\vec{E}_1 = f_1(z)\,\vec{u}_z$. Furthermore, the plane $z = \dfrac{h}{2}$ is a plane of symmetry for the charges, so in this plane $\vec{E}_1 = \vec{0}$ and two points symmetrical with respect to this plane have opposite fields.
For $0 < z < h$, $\vec{\nabla} \cdot \vec{E}_1 = \dfrac{\mathrm{d}f_1}{\mathrm{d}z} = \dfrac{\rho}{\epsilon}$ therefore $f_1(z) = \dfrac{\rho z}{\epsilon} + c$.
The field must be null in $z = \frac{h}{2}$, therefore $c = -\dfrac{\rho h}{2\epsilon}$ and $f_1(z) = \dfrac{\rho(z - h/2)}{\epsilon}$

For $z > h$, $\dfrac{\mathrm{d}f_1}{\mathrm{d}z} = 0$, therefore $f_1(z) = f_1(h) = \dfrac{\rho h}{2\epsilon}$ by continuity of $\vec{E}_1$ at $z = h$ (no surface charge density).

For $z < 0$, $f_1(h) = -\dfrac{\rho h}{2\epsilon}$ thanks to the symmetry with respect to the plane $z = h/2$ mentioned above.
The field $\vec{E}_2$ is identical to $\vec{E}_1$ except for the sign : $f_2(z) = -\dfrac{\rho(z - h/2)}{\epsilon}$ in the charged area, $-\dfrac{\rho h}{2\epsilon}$ when z approaches $+\infty$ et $+\dfrac{\rho h}{2\epsilon}$ when z approaches $-\infty$.
So outside the charged area, $\vec{E}_1 + \vec{E}_2 = \vec{0}$: the global electric field is nil;
and in the the charged area, $\vec{E} = \dfrac{\rho(z - h/2)}{\epsilon} - \dfrac{\rho(z + h/2)}{\epsilon}\,\vec{u}_z = \dfrac{\rho h}{\epsilon}\,\vec{u}_z$

Part 3 Useful formulae

Reminder of operator expressions :

In cylindrical coordinates :

Elementary displacement : $d\vec{OM} = dr\,\vec{u}_r + rd\theta\,\vec{u}_\theta + dz\,\vec{u}_z$

$$\vec{\nabla} U = g\vec{rad}\,U = \frac{\partial U}{\partial r}\vec{u}_r + \frac{1}{r}\frac{\partial U}{\partial \theta}\vec{u}_\theta + \frac{\partial U}{\partial z}\vec{u}_z$$

$$\vec{\nabla}\cdot\vec{E} = div\,\vec{E} = \frac{1}{r}\frac{\partial}{\partial r}(r\,E_r) + \frac{1}{r}\frac{\partial E_\theta}{\partial \theta} + \frac{\partial E_z}{\partial z}$$

$$\vec{\nabla}\times\vec{E} = r\vec{o}t\,\vec{E} = \left(\frac{1}{r}\frac{\partial E_z}{\partial \theta} - \frac{\partial E_\theta}{\partial z}\right)\vec{u_r} + \left(\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r}\right)\vec{u_\theta} + \left(\frac{1}{r}\frac{\partial}{\partial r}(r\,E_\theta) - \frac{1}{r}\frac{\partial E_r}{\partial \theta}\right)\vec{u_z}$$

In spherical Coordinates

Elementary displacement : $d\vec{OM} = dr\,\vec{u}_r + rd\theta\,\vec{u}_\theta + rsin(\theta)d\phi\,\vec{u}_\phi$

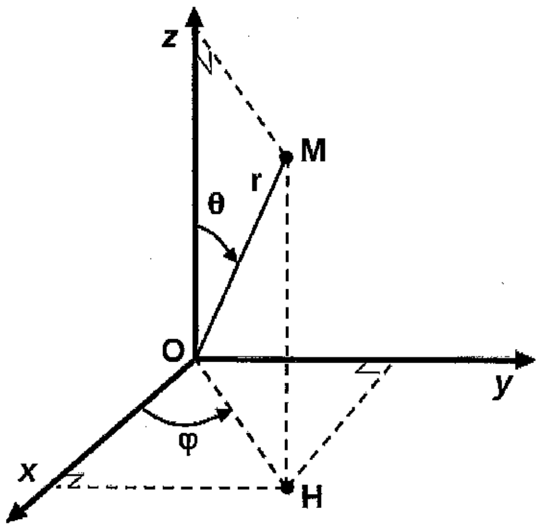

FIGURE 1: *Spherical coordinate system*

$$g\vec{rad}\,U = \vec{\nabla}U = \frac{\partial U}{\partial r}\vec{u}_r + \frac{1}{r}\frac{\partial U}{\partial \theta}\vec{u}_\theta + \frac{1}{rsin\,\theta}\frac{\partial U}{\partial \phi}\vec{u}_\phi$$

$$div\,\vec{E} = \vec{\nabla}\cdot\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\,E_r) + \frac{1}{r\,sin\,\theta}\frac{\partial}{\partial \theta}(sin\,\theta\,E_\theta) + \frac{1}{r\,sin\,\theta}\frac{\partial E_\phi}{\partial \phi}$$

$$r\vec{o}t\,\vec{E} = \vec{\nabla}\times\vec{E} = \left\{\frac{1}{rsin\theta}\left[\frac{\partial}{\partial \theta}(E_\phi sin\theta) - \frac{\partial E_\theta}{\partial \phi}\right]\right\}\vec{u_r} + \left\{\frac{1}{r}\left[\frac{1}{sin\theta}\frac{\partial E_r}{\partial \phi} - \frac{\partial}{\partial r}(rE_\phi)\right]\right\}\vec{u_\theta} + \left\{\frac{1}{r}\left[\frac{\partial}{\partial r}(rE_\theta) - \frac{\partial E_r}{\partial \theta}\right]\right\}\vec{u_\phi}$$