Exam n° 2 – 1 hour 30 minutes

- No documents, no calculators, no cell phones or electronic devices allowed.
- Take a deep breath before starting (everything is going to be ok!) and read entirely the exam before starting.[0]
- All exercises are independent, you can do them in the order that you'd like.
- Please start an exercise at the top of a page (for readability).
- Number single pages, or simply the booklets (*copies doubles*) if multiple : for example 1/3, 2/3, 3/3
- All your answers must be fully (but concisely) justified, unless noted otherwise.
- Redaction and presentation matter! For instance, write full sentences and make sure your 'x' and 'n' can be distinguished.
- **Respecting all of the above is part of the exam grade (0.5 points).** Provided rubric is indicative (changes may occur).

Warm-up exercises (8 points)

You are expected to provide some steps for those exercises. Little partial credit will be given for just writing the answer.

Exercise 1. True or False Justify briefly why the statement is True or False. If the statement is false, provide the correct answer (if applicable).

1. Given a polynomial $P \in \mathbb{R}_3[X]$, if 2 is root of P then $(X-2)$ divides P'.
2. By composition we have $\lim\limits_{x\to 0^+} 2\arctan(\ln(1-\sin(x))) = 0$.
3. Solving $\cos(x) = \sin(2x)$ over $\mathbb{R}$ leads to the set of solutions $S = \{\frac{\pi}{6}, \frac{\pi}{2}\} + \pi\mathbb{Z}$.

Exercise 2. Compute **ONE** of the following limits (your choice) :

$$\text{Choice A : } \lim_{x\to 0^+} \frac{e^{\cos(x)} - e}{\sin(x)}, \qquad \text{or} \qquad \text{Choice B : } \lim_{x\to+\infty} \left(1 + \sin\left(\frac{1}{x}\right)\right)^{2x}.$$

If you decide to do both and if one is incorrect, we will only consider the incorrect one (so choose, and choose wisely).

Exercise 3. Consider $f : A \to \mathbb{R}$ such that $f(x) = \dfrac{x^2 - x - 2}{18 + 2x^2 - 12x}$. Provide the domain of definition A. Sketch the graph of f (justify limit behaviors). Be as precise as possible in your answers.

Linear systems (5 points)

We expect **full details** on the steps, with proper operations and redaction, and a proper solution written in the end. If details not provided, we will not check your calculations.

Exercise 4. We consider the function $f : \mathbb{R}^4 \to \mathbb{R}^4$ defined by

$$f(x, y, z, t) = (x - t, 2x + y - 3t, 3x + 2y + z - 6t, 4x + 2y + 2z - 8t)$$

1. Compute $\ker(f)$.

2. What conditions do we have on the parameters a, b, c, d so that (a, b, c, d) admits (at least) a pre-image by f ?
3. Is f injective ? surjective ? Justify your answers.
4. Consider the system

$$(S)\begin{cases} 3x + 2y + z - 6t = 1 \\ 4x + 2y + 2z - 8t = 1 \\ x - t = 1 \\ 2x + y - 3t = 1 \end{cases}$$

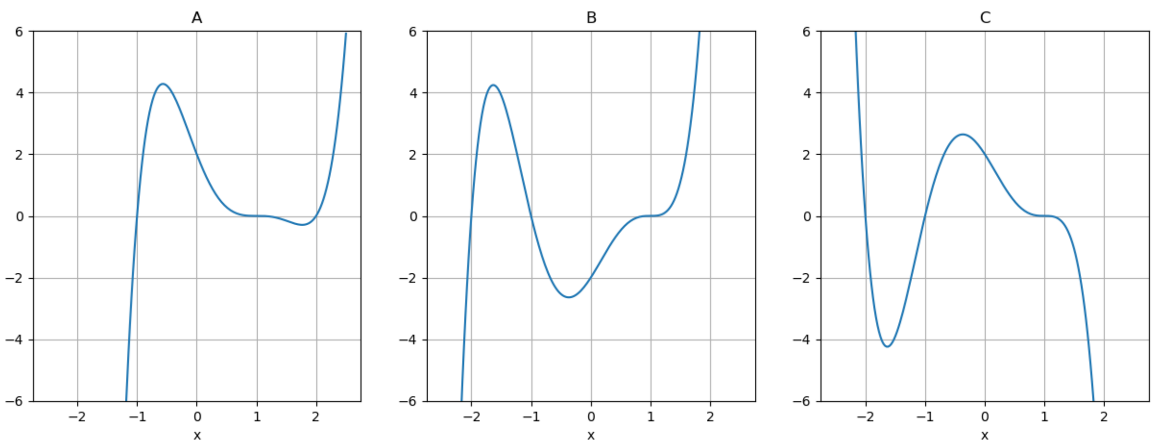
Does (S) have a (unique) solution ? Justify your answer.

Polynomials (4,5 points)

Exercise 5.

Let $a, b \in \mathbb{R}$, we consider the polynomial P defined by $P(X) = X^5 - 4X^3 + 2X^2 + aX + b$.

1. Find a, b such that 1 is root of P or multiplicity at least 2.
2. With the obtained choice of a, b, determine the multiplicity of the root 1.
3. Write the Taylor's formula for P at 1. We expect details on the coefficients computation.
4. Factorize P in $\mathbb{R}$.
5. Given the following graphs, which one corresponds to P ? Justify your answer.



Vector Subspace (3 points)

Exercise 6.

Consider the sets

$$F = \{(x, y, z) \in \mathbb{R}^3,\ x + y - 2z = 0\}, \quad G = \{(x, y, z) \in \mathbb{R}^3,\ x = 0\}.$$

1. Show that F, G are vector subspaces.
2. Write $F \cap G$. Is it a vector subspace ?
3. Provide a generating family of F, a generating family of G, and a generating family of $F \cap G$.
4. Is each of those families linearly independent ? What do you conclude ?

0. Draw a snowman next to your name on the first page once this is done.