End of First Semester Examination

Tuesday 31st January 2023 - Time Allowed: 3 hours

Instructions:
Make sure your work is ***well presented*** *and* ***readable****. Calculators and a hand-written formula sheet (one double-sided A4 page) allowed.* ***Un-justified answers may not be taken into account. The two exercises are independent.***

Formulae

Gradient in cylindrical polar coordinates:

$$\vec{\nabla} V = \overrightarrow{grad}(V) = \frac{\partial V}{\partial r}\vec{u_r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\vec{u_\theta} + \frac{\partial V}{\partial z}\vec{u_z}$$

Rotational in cylindrical polar coordinates:

$$\vec{\nabla} \wedge \vec{E} = \vec{rot}\ \vec{E} = \left(\frac{1}{r}\frac{\partial E_z}{\partial \theta} - \frac{\partial E_\theta}{\partial z}\right)\vec{u_r} + \left(\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r}\right)\vec{u_\theta} + \left(\frac{1}{r}\frac{\partial}{\partial r}(r\ E_\theta) - \frac{1}{r}\frac{\partial E_r}{\partial \theta}\right)\vec{u_z}$$

Total permittivity ε and free space permittivity ε_0 :

$$\varepsilon = \varepsilon_r \varepsilon_0\ ; \quad \varepsilon_0 = \frac{1}{36\pi} \times 10^{-9} F/m$$

Vector identity :

$$\vec{\nabla} \wedge (\vec{\nabla} \wedge \underline{\vec{E}}) = \vec{\nabla}(\vec{\nabla}.\underline{\vec{E}}) - \vec{\Delta}\underline{\vec{E}} \quad \text{or} \quad \vec{rot}(\vec{rot}(\underline{\vec{E}})) = \vec{grad}(div(\underline{\vec{E}})) - \vec{\Delta}\underline{\vec{E}}$$

Exercise 1: Non-ideal dielectric medium

In this exercise, we will study a parallel plate capacitor. In order to increase its capacity, we insert a dielectric material between the two conducting plates. Initially, we will consider an ideal dielectric material, then we will take two non-idealities encountered in real dielectric media into account which necessitate a more complex model of the capacitor.

1.1 Parallel plate capacitor with an ideal dielectric material

In this part, we will consider that a dielectric material with relative permittivity ε_r = 3.00 occupies the space between two conducting plates in the shape of disks, of axis (Oz), diameter d = 30.0 mm, and separated by a distance e = 0.100 mm. As e << d, we consider that the plates are infinite for the calculation of the electric field.

We first apply a fixed potential difference $U = V(z = e) - V(z = 0) = +1.00\ V$ between the two plates.

Question 1-1: Determine the topography of the field $\vec{E}$ in the dielectric medium. Give the literal expression for $\vec{E}$ as a function of the variables of the problem **by solving Maxwell's local (point form) equations** and give its numerical value.

Question 1-2: Using the volumetric density of electrostatic energy stored between the plates ($w_e = \frac{1}{2}\varepsilon E^2$), determine the literal expression and numerical value for the capacitance C_∞ of the capacitor with the dielectric material between the plates. Show that $\frac{C_\infty}{C_0} = \varepsilon_r$, where C_0 is the capacitance when there is vacuum in the space between the plates.

Question 1-3: With the help of a sketch, explain in a few sentences how the presence of electric dipoles in the dielectric medium between the plates leads to an increase in the capacitance of the capacitor. You must consider the static situation (U = constant) for this question, and you can compare two situations: one where the space between the capacitor plates is empty (vacuum), and the other where it is filled with a medium containing the afore mentioned dipoles.

1.2 First non-ideality: the dielectric material is not perfectly insulating

In this part, we will take into account a first non-ideality encountered in real dielectric media. We consider here that the dielectric medium of relative permittivity ε_r = 3.00 also has a low electrical conductivity $\gamma = 5.00 \times 10^{-14}$ S/m.

Question 1-4: The potential difference still being the same (U = +1.00 V), determine the literal expression for the volume current density $\vec{j}$ present between the two plates of the non-ideal capacitor in the static regime. Then, give the literal expression and the numerical value of the current *I* flowing between the two plates. Comment.

Question 1-5: Deduce from the previous question the expression and the numerical value of the leakage resistance R_f associated with the capacitor, as a function of γ, e, d.

*We will model this first non-ideality using an equivalent circuit composed of a **parallel combination of R_f and C_∞**. The voltage applied across the circuit is now **sinusoidal**, of angular frequency ω and amplitude $U_0 = 1.00V$.*

Question 1-6: Determine the complex impedance, denoted $\underline{Z_1}$, of this parallel combination as a function of R_f, C_∞ and ω.

*For the sake of simplicity, we can define the complex impedance $\underline{Z_1}$ as a **complex capacitance** $\underline{C_1}$ such that $\underline{Z_1} = \frac{1}{j\underline{C_1}\omega}$ and $\underline{C_1} = C_1' - jC_1''$.*

Question 1-7: Give the expressions of $\underline{C_1}$, C_1' and C_1''.

With this formalism, the relative permittivity is also complex $\underline{\varepsilon_r} = \varepsilon_r' - j\varepsilon_r''$, and is still defined as being the ratio of the capacitances with and without the presence of the dielectric medium: $\underline{\varepsilon_r} = \frac{\underline{C_1}}{C_0}$.

Question 1-8: Show that $\underline{\varepsilon_r} = \frac{C_\infty}{C_0} - \frac{j}{\omega R_f C_0}$.

Question 1-9: Give the expressions for ε_r' and ε_r''. Then, after a study of the behaviors at limiting frequencies, sketch and comment on the variation of ε_r' and ε_r'' as a function of the angular frequency ω. In what range of angular frequencies is the conductivity, through ε_r'', dominant?

1.3 Second non-ideality: dipolar relaxation phenomenon

*At high angular frequencies, when $\omega \geq \omega_d$, the dipoles are not able to follow the oscillation rate of the electric field. We model this phenomenon, called "dielectric relaxation", by **the association of a resistance R_d and a capacitance C_d in series**. Once again, we can consider this association in series as a **complex impedance $\underline{Z_2} = \frac{1}{j\underline{C_2}\omega}$**.*

Question 1-10: Give the literal expressions for $\underline{C_2}$ as a function of R_d, C_d and ω. (Note: in this question we will consider only the mesh that contains R_d and C_d).

Now, this association of R_d and C_d in series is added in parallel to the previous circuit, as shown in Figure 1.

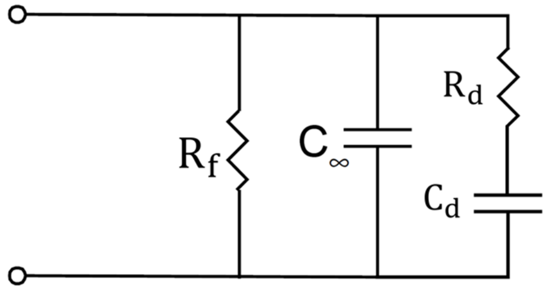

Figure 1: Non ideal capacitor's equivalent electric circuit, containing the leakage resistance and dipolar relaxation.

Question 1-11: Considering the circuit as a whole – the association of $\underline{C_1}$ and $\underline{C_2}$ in parallel – as a complex capacitance $\underline{C_{tot}}$, show that

$$\underline{\varepsilon_{r,tot}}(\omega) = \varepsilon_{r,\infty} + \frac{\Delta}{1+j\omega\tau_d} - \frac{j}{\omega R_f C_0},$$

detailing the expressions of $\varepsilon_{r,\infty}$, Δ and τ_d as a function of C_∞, C_0, R_d and C_d. (We remind you that $\underline{\varepsilon_{r,tot}} = \frac{\underline{C_{tot}}}{C_0}$).

Figure 2 shows a measurement realized on a dielectric material under consideration for the construction of a capacitor of a large capacitance.

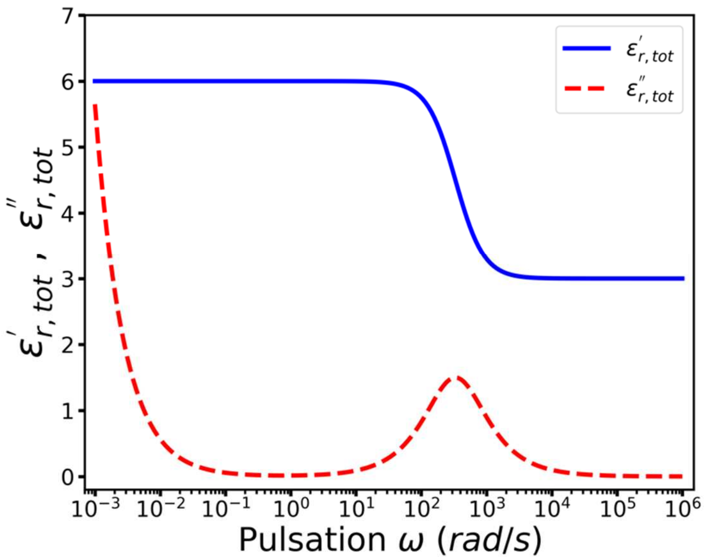

Figure 2: Real ($\varepsilon'_{r,tot}$) and imaginary ($\varepsilon''_{r,tot}$) parts of the complex permittivity of a non-ideal dielectric material, as a function of the angular frequency ω ("pulsation", in French).

Question 1-12: Using the complex expression from the previous question (but without transforming it), study the behavior of the real and imaginary parts of

$$\underline{\varepsilon_{r,tot}}(\omega) = \varepsilon'_{\text{r,tot}}(\omega) - j\varepsilon''_{\text{r,tot}}(\omega),$$

at their limits when $\omega \longrightarrow 0$, $\omega \longrightarrow +\infty$ and $\omega = \frac{1}{\tau_d}$. Justify that $\omega = \frac{1}{\tau_d}$ corresponds to the local maximum of the ε''_r curve. We will assume that $\tau_d \ll R_f C_0$. Then, determine the numerical values of $\varepsilon_{r,\infty}$ and Δ using Figure 2.

Question 1-13: Using Figure 2, determine the range(s) of angular frequencies ω for which you think this material would be interesting for the application at hand? Would you use this material to construct a capacitor that works at 50 Hz (angular frequency of 100π rad/s)? Justify your answer.

Exercise 2: Influence of the frequency on electromagnetic induction phenomena in a cable crossed by a current

The two parts of this exercise can be treated independently one from another.

We have seen in tutorials that if we consider a plane monochromatic electromagnetic wave of angular frequency ω (ω being in the range where the quasi-static approximation applies), this wave propagates in the conductor over a distance of a few times δ, where:

$$\delta = \sqrt{\frac{2}{\mu_0 \gamma \omega}}$$

given that μ_0 is vacuum permeability and γ the electrical conductivity.

The parameter δ is called the ***skin depth*** *or* ***penetration depth of the electromagnetic waves in the conductor****. The objective of this exercise is* ***to prove that δ is also the penetration depth of induction phenomena in a metal****. We will also assess to what extent the electromagnetic induction phenomenon influences the conduction in a copper wire (or cable) of radius a, when the penetration depth δ remains small with respect to a.*

From now on, we consider a straight, infinite, homogeneous copper wire of axis Oz and radius a, which is a good conductor ($\gamma = 5.10^7 S.m^{-1}$) of permeability $\mu_0=4.\pi.10^{-7}$ H/m. We assume that a=1.5mm.

2.1 First part: at low frequency the induced currents influence only weakly the wire's conduction

The wire is connected to a voltage source: consequently, it is crossed by a constant current of intensity I associated with a uniform volume current density: $\vec{J} = J_0\,\vec{u}_z$

Question 2-1: Calculate *I* as a function of J_0 and *a*.

Question 2-2: Establish in a detailed manner the expression of the magnetic field $\vec{B}$ (direction, orientation, modulus) created in the whole space by this DC current. The result will be expressed as a function of μ_0, J_0 and of the cylindrical coordinates of axis Oz to express the position of a given point *M*. You will have to distinguish the cases where *M* is inside ($r<a$) or outside ($r>a$) the conducting wire.

From now on, we are only interested in the ***region inside the conductor****. Considering a low frequency, if the current intensity becomes sinusoidal, we can consider in the framework of the quasi-static approximation that:*

- *The conductivity remains the same as when considering a direct (DC) current*
- *The volume current density J remains uniform: $\vec{J} = J_0 cos(\omega t).\vec{u}_z$*
- *The expression of the magnetic field $\vec{B}$ is: $\overrightarrow{B_0}(r).cos(\omega t)$ with $\overrightarrow{B_0}(r)$ calculated before.*

Question 2-3: What is Maxwell's equation related to electromagnetic induction? Justify with the help of this equation that an induced electromagnetic field $\vec{E}_{ind}$ comes up in the conductor.

We will assume that $\vec{E}_{ind}$ is along the Oz direction and nil on the Oz axis[1]*. Note that $\vec{E}_{ind}$ depends only on r and t and is the source of an induced current $\vec{J}_{ind}$ which adds to the preceding volume current density $\vec{J}$.*

Question 2-4: In a general way, when considering a conductor under the quasi-static assumption, one contribution to the total current is neglected. What is this contribution? Justify this choice with a calculation based on the magnitudes of the two current contributions.

Question 2-5: Starting with the result of Question 2-3, for $r<a$ determine the literal expression of $\vec{E}_{ind}$ and show that its modulus is:

[1] Note that there is no direct similarity between this exercise and the subject 8 from the Tutorials given that the conductors' nature and configuration are totally different

$$\| \vec{E}_{ind} \| = \frac{\mu_0 . J_0 \omega}{4} . \sin(\omega t) . r^2$$

This field generates an induced current $\vec{J}_{ind} = \gamma . \vec{E}_{ind}$

Question 2-6: Express $\vec{J}_{ind}$ as a function of J_0, ω, δ and space and time variables. To achieve this, you can get back to the definition of the penetration depth δ. For which value of r is the amplitude of $\|\vec{J}_{ind}\|$ maximum in the conductor? We will call this value $J_{ind,max}$. Calculate the ratio $\frac{J_{ind,max}}{J_0}$ and do the numerical application for a=1.5mm and f=50Hz. Draw conclusions on the influence of induced currents at this frequency.

2.2 Second part: induced currents as a dominating phenomenon

For a wire having a given radius a, ***still being under the quasi-static approximation****, if we consider that the frequency increases, the* ***contribution of the induced currents becomes huge compared to that of the conduction current*** *:* $\vec{J} = J_0 cos(\omega t) . \vec{u}_z$ *. Due to this, both Maxwell-Ampere and Maxwell-Faraday relationships have to be used to solve the problem, that is, to calculate* J_{ind}.

From now on, we will use the complex notation $\underline{\vec{E}_{ind}}$ *simply denoted:* $\underline{\vec{E}} = \underline{E}(r,t)\ e^{j\omega t}\ \vec{u}_z$
as only the ***induced electric field*** *is taken into account in this part. The* ***conductor is assumed to be uncharged****.*

Question 2-7: Starting with Maxwell's equations in a way analogous to the calculation of the equation of propagation of the waves in a conductor (subject 7) prove that the differential equation fulfilled by $\underline{\vec{E}}$ is:

$$\vec{\Delta}\underline{\vec{E}} = \frac{2j}{\delta^2}\ \underline{\vec{E}}$$

(j is such that $j^2 = -1$).

Indication: you can use the fact that the displacement current can be neglected and get back to the definition of the penetration depth.

In cylindrical coordinates and in the conditions of our problem, it is given that the scalar equation containing $\underline{E}$(r) becomes:

$$\frac{\partial^2 \underline{E}(r)}{\partial r^2} = \frac{2j}{\delta^2}\underline{E}(r)$$

Question 2-8: We make the hypothesis that the solution of this differential equation is of the type:

$$\underline{\vec{E}}(r,t) = K_1 e^{\frac{\underline{\alpha} r}{\delta}}\, e^{j\omega t}\ \vec{u}_z$$

where K_1 is a real constant and $\underline{\alpha}$ a complex constant (with no unit).

Prove that the values of $\underline{\alpha}$ which satisfy the wave equation are: $\underline{\alpha} = \pm(1+j)$. Establish the real expression *E(r,t).* Make a sketch to represent in the two cases how the wave propagates in the cable and completely characterize them (give also their phase velocity V).

In the following, we take into account the fact that the outer part of the cable (r close to a) has a higher conduction than its core (r close to 0), which was one of the conclusions of the 1st part of the exercise. Consequently, we admit as a ***possible solution only the wave whose amplitude increases with r.***
The induced current density is then: $\vec{J}_{ind} = J(r,t) . \vec{u}_z$

Question 2-9: Plot first the **envelope function** of *J(r,t)* as a function of *r* for *t=2T*. The frequency being set to 800 kHz, calculate literally and numerically the wavelength λ. Complete your preceding graph by plotting the function *J(r, 2T)* and pay a particular attention to its value when *r=a*. Comment on the graph.

Question 2-10: Assume that the frequency of the signal that propagates in the wire increases. Which consequence can the penetration depth have on the wire resistance?