SCAN 2 - Test # 3 - Correction - Feb. 4th, 2025

Exercise 1

All questions in this exercise are independent.

1. Determine the nature of the following two series and integral :

(a) $\displaystyle\sum_{n\geqslant 1}\ln\left(1+\frac{1}{n^4+2n}\right)$ (b) $\displaystyle\sum_{n\geqslant 0}\frac{\sqrt{n}e^{n^2}}{n!}$ (c) $\displaystyle\int_1^{+\infty}\frac{e^{-x}(x^3+2)}{\sqrt{x}}\,dx$

2. Consider the following integral :

$$\mathcal{I}=\int_1^{+\infty}\frac{1}{x\sqrt{1+x^2}}\,dx.$$

Using the substitution $t=\sqrt{1+x^2}$, compute $\mathcal{I}$.

1) a) Let $u_n = \ln\left(1+\dfrac{1}{n^4+2n}\right)$. $u_n \underset{+\infty}{\sim} \dfrac{1}{n^4} \geqslant 0$. $\displaystyle\sum_{n\geqslant 1}\frac{1}{n^4}$ converges (Riemann series)

Hence $\displaystyle\sum_{n\geqslant 1} u_n$ converges (equivalent test).

b) Let $u_n = \dfrac{\sqrt{n}\,e^{n^2}}{n!} \geqslant 0$. Ratio test :

$\forall n \geqslant 1$, $\dfrac{u_{n+1}}{u_n} = \dfrac{\sqrt{n+1}\,e^{(n+1)^2}}{(n+1)!}\cdot\dfrac{n!}{\sqrt{n}\,e^{n^2}} = \sqrt{1+\dfrac{1}{n}}\cdot\dfrac{e^{2n+1}}{n+1} \xrightarrow[n\to+\infty]{} +\infty$

Hence $\displaystyle\sum_{n\geqslant 1} u_n$ CV. (comparative growth)

c) Let $f(x) = \dfrac{e^{-x}(x^3+2)}{\sqrt{x}}$ for $x>0$.

Note that $\displaystyle\lim_{x\to+\infty} x^2. f(x) = 0$ (comparative growth).

Hence $f(x) \underset{x\to+\infty}{=} o\left(\dfrac{1}{x^2}\right)$, $f(x)\geqslant 0$. Since $\displaystyle\int_1^{+\infty}\frac{1}{x^2}dx$ CV,

$\displaystyle\int_1^{+\infty} f(x)\,dx$ CV due to the comparison test.

2) Let $I = \displaystyle\int_1^L \frac{1}{x\sqrt{1+x^2}}dx$, $L>1$.

substitution : ① $t = \sqrt{1+x^2}$

② $dt = \dfrac{\cancel{2}x}{\cancel{2}\sqrt{1+x^2}}dx \iff \dfrac{1}{x^2}dt = \dfrac{1}{x\sqrt{1+x^2}}dx$

Since $t\geqslant 0$, $x^2 = t^2-1$ hence $\dfrac{dx}{x\sqrt{1+x^2}} = \dfrac{dt}{t^2-1}$.

③ Boundaries. $\begin{cases} x=1 \iff t=\sqrt{2} \\ x=L \iff t=\sqrt{1+L^2}\end{cases}$.

Hence : $I = \displaystyle\int_{\sqrt{2}}^{\sqrt{1+L^2}}\frac{dt}{t^2-1}$.

Also : $\dfrac{1}{t^2-1} = \dfrac{a}{t-1}+\dfrac{b}{t+1}$ (partial fraction decomposition)

$a = \displaystyle\lim_{t\to1}\frac{1}{t+1} = \frac{1}{2}$ and $b = \displaystyle\lim_{t\to-1}\frac{1}{t-1} = -\frac{1}{2}$.

Thus, $I = \displaystyle\int_{\sqrt{2}}^{\sqrt{1+L^2}}\left(\frac{1}{2}\cdot\frac{1}{t-1}-\frac{1}{2}\cdot\frac{1}{t+1}\right)dt = \frac{1}{2}\Big[\ln|t-1|-\ln|t+1|\Big]_{\sqrt{2}}^{\sqrt{1+L^2}}$

$= \dfrac{1}{2}\ln\left|\dfrac{\sqrt{1-L^2}-1}{\sqrt{1-L^2}+1}\right| - \dfrac{1}{2}\ln\left|\dfrac{\sqrt{2}-1}{\sqrt{2}+1}\right| = \dfrac{1}{2}\ln\left|\dfrac{1-\frac{1}{\sqrt{1-L^2}}}{1+\frac{1}{\sqrt{1-L^2}}}\right| - \dfrac{1}{2}\ln\left(\dfrac{\sqrt{2}-1}{\sqrt{2}+1}\right) \xrightarrow[L\to+\infty]{} -\dfrac{1}{2}\ln\left(\dfrac{\sqrt{2}-1}{\sqrt{2}+1}\right)$

Exercise 2

Consider a sequence $(u_n)_{n\in\mathbb{N}}$ defined by $u_0 \in]0,1[$ and the following iterative relation :

$$u_{n+1} = \frac{2u_n}{u_n+2}.$$

In this exercise, we will determine the nature of the series $\sum_n u_n$.

1. (a) Prove that $u_n \in]0,1[$ for all $n \in \mathbb{N}$.

   (b) Show that the sequence $(u_n)_{n\in\mathbb{N}}$ is decreasing.

2. Deduce that the sequence $(u_n)_{n\in\mathbb{N}}$ converges to a limit ℓ, and determine that limit.

3. Simplify the sum $\sum_{n=0}^{N} \ln\left(\frac{u_n}{u_{n+1}}\right)$ then deduce that the series $\sum_{n\geqslant 0} \ln\left(\frac{u_n}{u_{n+1}}\right)$ diverges.

4. Show that $\ln\left(\frac{u_n}{u_{n+1}}\right) \underset{n\to+\infty}{\sim} \frac{u_n}{2}$ then deduce the nature of $\sum_n u_n$.

1) a) Induction proof:
- $u_0 \in]0,1[$
- induction hypothesis: $u_n \in]0,1[$ for an $n \in \mathbb{N}$.

$0 < u_n < 1 \Rightarrow 0 < 2u_n < 2$ and $\frac{1}{3} < \frac{1}{u_n+2} < \frac{1}{2}$

hence $0 < \frac{2u_n}{u_n+2} < 1$: $u_{n+1} \in]0,1[$.

b) Let $n \in \mathbb{N}$. $u_{n+1} - u_n = \frac{2u_n}{u_n+2} - u_n = \frac{-u_n}{u_n+2} < 0$: (u_n) is decreasing.

2) (u_n) is bounded from below by 0, and is decreasing.
Monotone CV theorem: (u_n) converges to $l \geqslant 0$. f being continuous, its limit is a fixed point of f : $f(l) = l \Leftrightarrow \frac{2l}{l+2} = l$

$\Leftrightarrow l^2 = 0$

$\Leftrightarrow \boxed{l=0}$

3) Let $N \in \mathbb{N}$.
- $\sum_{n=0}^{N} \ln\left(\frac{u_n}{u_{n+1}}\right) = \sum_{n=0}^{N} \ln(u_n) - \ln(u_{n+1}) = \ln(u_0) - \ln(u_{N+1})$ (telescoping sum).
- $\lim_{N\to+\infty} \sum_{n=0}^{N} \ln\left(\frac{u_n}{u_{n+1}}\right) = \ln(u_0) - \lim_{N\to+\infty} \ln(u_{N+1}) = +\infty$. since $\lim_{+\infty} u_n = 0$

Hence the series $\sum_{n\geqslant 0} \ln\left(\frac{u_n}{u_{n+1}}\right)$ DV.

4) Let $n \in \mathbb{N}$. $\frac{u_n}{u_{n+1}} = \frac{u_n}{2u_n} \cdot (u_n+2) = 1 + \frac{u_n}{2}$

since $\ln(1+x) \underset{x\to 0}{\sim} x$, $\ln\left(1+\frac{u_n}{2}\right) \underset{n\to+\infty}{\sim} \frac{u_n}{2}$.

We know that $\sum_{n\geqslant 1} \ln\left(\frac{u_n}{u_{n+1}}\right)$ DV. With the equivalent test ($\geqslant 0$ terms) we deduce that $\sum_n \frac{u_n}{2}$ DV, hence $\sum_n u_n$ DV

Exercise 3

For any real number a, consider M_a as the matrix in $\mathcal{M}_3(\mathbb{R})$ defined by $M_a = \begin{pmatrix} 1 & 0 & 4 \\ a & 2 & 0 \\ 2 & 0 & 3 \end{pmatrix}$.

Part I :

1. Show that $u = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector of M_a, specifying the associated eigenvalue.
2. Show that for any value of $a \in \mathbb{R}$, the matrix M_a is diagonalisable.
3. Are there values of a for which the matrix M_a is invertible ?
4. We now seek functions x, y, and z of class $\mathcal{C}^1$ on $\mathbb{R}$ that satisfy the following system :

$$(\mathcal{S}) \quad \begin{cases} x'(t) = x(t) + 4z(t) \\ y'(t) = ax(t) + 2y(t) \\ z'(t) = 2x(t) + 3z(t) \end{cases}$$

and the initial conditions $x(0) = 3$, $y(0) = 1$, and $z(0) = 0$. We define $X(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$ for all $t \in \mathbb{R}$. Notice that system $(\mathcal{S})$ can be rewritten as $X'(t) = M_a X(t)$.

(a) Find a diagonal matrix D_1 and an invertible matrix P such that $M_a = PD_1P^{-1}$. It is not necessary to determine P^{-1}.

(b) Let $Y(t) = P^{-1}X(t)$ with $Y(t) = \begin{pmatrix} \alpha(t) \\ \beta(t) \\ \gamma(t) \end{pmatrix}$. Show that $X'(t) = M_a X(t)$ if and only if $Y'(t) = D_1 Y(t)$.

(c) Solve the system $Y'(t) = D_1 Y(t)$.

(d) Determine $X(t)$ using the initial conditions.

1) $M_a\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ hence $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is an eigvect associated to the eigval $\lambda = 2$.

2) $P_{M_a}(\lambda) = \begin{vmatrix} 1-\lambda & 0 & 4 \\ a & 2-\lambda & 0 \\ 2 & 0 & 3-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = (2-\lambda)(\lambda^2 - 4\lambda - 5)$
$= (2-\lambda)(\lambda-5)(\lambda+1)$.

$\forall a \in \mathbb{R}$, M_a has 3 different eig. vals 2, 5 and -1, of multiplicity 1.
Hence M_a is diagonalisable.

3) From $P_{M_a}(\lambda)$ we deduce $\det(M_a) = 2 \times (-5) \neq 0$
Hence $\forall a \in \mathbb{R}$, M_a is invertible.
or: $0 \notin Sp(M_a)$ so M_a is invertible. (property of the eig. val 0)

4) a) $D_1 = \text{Diag}(-1, 2, 5)$ and P is a matrix which colums are 3 eigvects of D with respect to $-1, 2, 5$.

- $E_{-1} = \text{Ker}(M_a + I)$

$M_a + I = \begin{pmatrix} 2 & 0 & 4 \\ a & 3 & 0 \\ 2 & 0 & 4 \end{pmatrix}$ Let $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$(M_a + I)X = 0 \iff \begin{cases} 2x + 4z = 0 \\ ax + 3y = 0 \end{cases} \iff \begin{cases} z = -\frac{1}{2}x \\ y = -\frac{a}{3}x \end{cases}$

Hence $v_{-1} = \begin{pmatrix} 6 \\ -2a \\ -3 \end{pmatrix} \in E_{-1}$. $E_{-1} = \text{Vect}(v_{-1})$ since $\dim E_{-1} = 1$.

- E_2: we know from 1) that $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in E_2$ hence $E_2 = \mathrm{Vect}(v_2)$.

- E_5: $M_a - 5I = \begin{pmatrix} -4 & 0 & 4 \\ a & -3 & 0 \\ 2 & 0 & -2 \end{pmatrix}$

$$X \in E_5 \iff (M_a - 5I)X = 0 \iff \begin{cases} -4x + 4z = 0 \\ ax - 3y = 0 \\ 2x - 2z = 0 \end{cases}$$

$$\iff \begin{cases} z = x \\ y = \frac{a}{3}x \end{cases}$$

$v_5 = \begin{pmatrix} 3 \\ a \\ 3 \end{pmatrix} \in E_5$

since $\dim(E_5) = 1$, $E_5 = \mathrm{Vect}(v_5)$.

We then deduce that $P = \begin{array}{c} \begin{matrix} v_1 & v_2 & v_5 \\ \downarrow & \downarrow & \downarrow \end{matrix} \\ \begin{pmatrix} 6 & 0 & 3 \\ -2a & 1 & a \\ -3 & 0 & 3 \end{pmatrix} \end{array}$ verifies $M_a = PD_1P^{-1}$

Remark: any proportional column is correct too.

b) Let $Y(t) = P^{-1}X(t)$, for $t \in \mathbb{R}$.

$$X'(t) = M_a X(t) \iff X'(t) = (PD_1P^{-1})X(t) \quad \big\downarrow \times P^{-1} \text{ on the left}$$

$$\iff \underbrace{P^{-1}X'(t)}_{Y'(t)} = D_1(\underbrace{P^{-1}X(t)}_{Y(t)})$$

$$\iff Y'(t) = D_1 Y(t).$$

c) $$Y'(t) = D_1 Y(t) \iff \begin{cases} \alpha'(t) = -\alpha(t) \\ \beta'(t) = 2\beta(t) \\ \gamma'(t) = 5\gamma(t) \end{cases}$$

$$\iff \begin{cases} \alpha(t) = C_1 e^{-t} \\ \beta(t) = C_2 e^{2t} \\ \gamma(t) = C_3 e^{5t} \end{cases}$$ where $C_1, C_2, C_3 \in \mathbb{R}$.

d) The relation $X(t) = PY(t)$ yields:

$$\begin{cases} x(t) = 6C_1 e^{-t} + 3C_3 e^{5t} \\ y(t) = -2aC_1 e^{-t} + C_2 e^{2t} + aC_3 e^{5t} \\ z(t) = -3C_1 e^{-t} + 3C_3 e^{5t} \end{cases}$$

The initial condition $\begin{cases} x(0) = 3 \\ y(0) = 1 \\ z(0) = 0 \end{cases}$ gives $\begin{cases} 6C_1 + 3C_3 = 3 \\ -2aC_1 + C_2 + aC_3 = 1 \\ -3C_1 + 3C_3 = 0 \end{cases}$ (S).

$$(S) \iff \begin{cases} C_1 = C_3 = \frac{1}{3} \\ -2aC_1 + C_2 + aC_3 = 1 \end{cases} \iff \begin{cases} C_1 = 1/3 \\ C_2 = 1 + \frac{1}{3}a \\ C_3 = 1/3 \end{cases}.$$

Finally: $$\begin{cases} x(t) = 2e^{-t} + e^{5t} \\ y(t) = -\frac{2}{3}a e^{-t} + \left(1 + \frac{a}{3}\right)e^{2t} + \frac{a}{3}e^{5t} \\ z(t) = -e^{-t} + e^{5t} \end{cases}.$$

Part II : We consider two real sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ satisfying the following system :

$$\begin{cases} u_{n+1} = u_n + 4v_n \\ v_{n+1} = 2u_n + 3v_n \end{cases}$$

with $u_0, v_0 \in \mathbb{R}$. Moreover, we define $U_n = \begin{pmatrix} u_n \\ v_n \end{pmatrix}$ for all $n \in \mathbb{N}$.

1. Determine the matrix N such that $U_{n+1} = NU_n$ for all $n \in \mathbb{N}$.
2. Express U_n in terms of U_0, N, and n.
3. (a) Determine a diagonal matrix D_2 and an invertible matrix Q such that $N = QD_2Q^{-1}$, then compute Q^{-1}.

   (b) Show that $N^n = QD_2^nQ^{-1}$ for all $n \in \mathbb{N}$.

   (c) Deduce u_n and v_n as functions of n, u_0, and v_0.
4. We now define $V_n = \begin{pmatrix} u_n \\ w_n \\ v_n \end{pmatrix}$ for all $n \in \mathbb{N}$. Using the previous questions, determine the expressions of the general terms of the sequences $(u_n)_{n\in\mathbb{N}}$, $(w_n)_{n\in\mathbb{N}}$, and $(v_n)_{n\in\mathbb{N}}$ satisfying the relation $V_{n+1} = M_0V_n$, in terms of n, u_0, v_0, and w_0.

1) $N = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$

2) Let $n \in \mathbb{N}$. $U_n = NU_{n-1} = N^2U_{n-2} = \dots = N^nU_0$.

3) a) $\det(N - \lambda I_2) = \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 5 = (\lambda+1)(\lambda-5)$.

$Sp(N) = \{-1, 5\}$. N is diagonalisable since it has 2 real eigvals.

E_5 : $(N - 5I)X = 0 \iff -4x + 4y = 0 \iff x = y$

$\Rightarrow E_5 = \mathrm{Vect}((1,1))$

E_{-1} : $(N + I)X = 0 \iff 2x + 4y = 0 \iff x = -2y$

$\Rightarrow E_{-1} = \mathrm{Vect}((-2,1))$

Hence, $D_2 = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$ and $Q = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}$.

Computing Q^{-1} we find $Q^{-1} = \frac{1}{3}\begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}$.

b) Since $N = QD_2Q^{-1}$, we deduce for $n \in \mathbb{N}$:

$$N^n = \underbrace{(QD_2\underbrace{Q^{-1})(Q}_{I_2}D_2\underbrace{Q^{-1}) \dots (Q}_{I_2}D_2Q^{-1})}_{n\text{ times}} = QD_2^nQ^{-1}$$

c) We know that $U_n = N^nU_0$:

- $N^n = \frac{1}{3}\begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} (-1)^n & 0 \\ 0 & 5^n \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2(-1)^n + 5^n & -2(-1)^n + 2.5^n \\ (-1)^{n+1} + 5^n & (-1)^n + 2.5^n \end{pmatrix}$

- computing $U_n = N^nU_0$ we find

$\forall n \in \mathbb{N}$, $\begin{cases} u_n = \frac{1}{3}\left((2(-1)^n + 5^n)u_0 + (-2(-1)^n + 2.5^n)v_0\right) \\ v_n = \frac{1}{3}\left(((-1)^{n+1} + 5^n)u_0 + ((-1)^n + 2.5^n)v_0\right). \end{cases}$

4) Notice that $V_{n+1} = M_0V_n \iff \begin{cases} u_{n+1} = u_n + 4v_n & (1) \\ w_{n+1} = 2w_n & (2) \\ v_{n+1} = 2u_n + 3v_n & (3) \end{cases}$

We already solved $\begin{cases} (1) \\ (3) \end{cases}$. Solving (2) yields $w_n = 2^nw_0$ (geometric sequence)

So we find $\begin{cases} u_n = (\text{same as before}) \\ w_n = 2^nw_0 \\ v_n = (\text{same as before}). \end{cases}$

Exercice 4

We recall that $1+2+\ldots+n = \dfrac{n(n+1)}{2}$.
Let $n \in \mathbb{N}^\star$. For all $x \in \mathbb{R}$, we define

$$A(x) = \begin{pmatrix} 1+x & 1 & \ldots & 1 \\ 2 & 2+x & \ldots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ n & n & \ldots & n+x \end{pmatrix}.$$

1. Compute $\det(A(x))$.
2. We aim to retrieve the result from the previous question by directly evaluating the eigenvalues of $A(x)$.
   (a) Determine the rank of $A(0)$.
   (b) Deduce that for all $x \in \mathbb{R}$, $A(x)$ has an eigenvalue with multiplicity at least $n-1$, which depends on x and will be specified.
   (c) Show that $\lambda = x + \dfrac{n(n+1)}{2}$ is an eigenvalue of $A(x)$.
   (d) Deduce from (b) and (c) the value of $\det(A(x))$.

1) Let $x \in \mathbb{R}$. $\det A(x) = \begin{vmatrix} 1+x & 1 & \cdots & 1 \\ 2 & 2+x & & 2 \\ \vdots & \vdots & & \vdots \\ n & n & & n+x \end{vmatrix} \underset{L_n \leftarrow \sum_{i=1}^{n} L_i}{=} \left(\dfrac{n(n+1)}{2}+x\right) \begin{vmatrix} 1+x & 1 & \cdots & 1 \\ 2 & 2+x & \cdots & 2 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{vmatrix}$

$$\underset{\substack{L_1 \leftarrow L_1 - L_n \\ L_2 \leftarrow L_2 - 2L_n \\ \vdots \\ L_{n-1} \leftarrow L_{n-1} - (n-1)L_n}}{=} \left(\dfrac{n(n+1)}{2}+x\right) \underbrace{\begin{vmatrix} x & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ 0 & \cdots & x & 0 \\ 1 & \cdots & 1 & 1 \end{vmatrix}}_{\text{diagonal matrix}}$$

$$= x^{n-1}\left(\dfrac{n(n+1)}{2}+x\right)$$

2) a) All the columns of $A(0)$ are the same
$\Rightarrow rk(A(0)) = 1$.

b) $\forall x \in \mathbb{R}$, $A(x) = A(0) + xI_n \Rightarrow A(0) = A(x) - xI_n$.
Hence, $\dim(\operatorname{Ker}(A(0))) = \dim(\operatorname{Ker}(A(x) - xI_n))$
Rank theorem: $\dim(\operatorname{Ker}(A(0))) + \underbrace{rk(A(0))}_{1} = \dim(\mathbb{R}^n) = n$

$\Rightarrow \dim(\operatorname{Ker}(A(0))) = n-1$
$\Rightarrow \dim(\operatorname{Ker}(A(x) - xI_n)) = \dim(E_x) = n-1$

A property says that if m is the multiplicity of an eigenvalue λ, then $1 \leq \dim(E_\lambda) \leq m$. Applying it for x, we get :
$\Rightarrow$ x is an eigenvalue of multiplicity $\geq n-1$.

c) We know that $tr(A(x))$ is equal to the sum of all $A(x)$'s eigenvalues, counted with their multiplicity : denote $\lambda \in \mathbb{R}$ the last eigval, we get $tr(A(x)) = (n-1)x + \lambda$. (1)
On the other hand, $tr(A(x)) = \dfrac{n(n+1)}{2} + nx$. (2)
(1) $-$ (2) : $\lambda = x + \dfrac{n(n+1)}{2}$.

d) $\det(A(x))$ is the product of all $A(x)$'s eigvals.
$\det(A(x)) = x^{n-1}\left(x + \dfrac{n(n+1)}{2}\right)$.