INSA LYON

FIMI - SCAN 2

Math Test # 3 on February 4th, 2025 - Duration 2h

Warnings and Advice

- All documents, dictionnaries, calculators or electronic devices, and communication means are prohibited.
- The marking scheme is provided for reference only.
- Presentation, quality of writing, clarity, and precision of reasoning are taken into account in the grading.
- Indicative grading scale : 4, 3.5, 8.5, 4

Exercise 1

All questions in this exercise are independent.

1. Determine the nature of the following two series and integral :

$$\text{(a)} \quad \sum_{n\geqslant 1} \ln\left(1+\frac{1}{n^4+2n}\right) \qquad \text{(b)} \quad \sum_{n\geqslant 0} \frac{\sqrt{n}\mathrm{e}^{n^2}}{n!} \qquad \text{(c)} \quad \int_1^{+\infty} \frac{\mathrm{e}^{-x}(x^3+2)}{\sqrt{x}}\,\mathrm{d}x$$

2. Consider the following integral :

$$\mathcal{I} = \int_1^{+\infty} \frac{1}{x\sqrt{1+x^2}}\,\mathrm{d}x.$$

   Using the substitution $t = \sqrt{1+x^2}$, compute $\mathcal{I}$.

Exercise 2

Consider a sequence $(u_n)_{n\in\mathbb{N}}$ defined by $u_0 \in]0,1[$ and the following iterative relation :

$$u_{n+1} = \frac{2u_n}{u_n+2}.$$

In this exercise, we will determine the nature of the series $\sum_n u_n$.

1. (a) Prove that $u_n \in]0,1[$ for all $n \in \mathbb{N}$.

   (b) Show that the sequence $(u_n)_{n\in\mathbb{N}}$ is decreasing.

2. Deduce that the sequence $(u_n)_{n\in\mathbb{N}}$ converges to a limit ℓ, and determine that limit.

3. Simplify the sum $\sum_{n=0}^{N} \ln\left(\frac{u_n}{u_{n+1}}\right)$ then deduce that the series $\sum_{n\geqslant 0} \ln\left(\frac{u_n}{u_{n+1}}\right)$ diverges.

4. Show that $\ln\left(\frac{u_n}{u_{n+1}}\right) \underset{n\to+\infty}{\sim} \frac{u_n}{2}$ then deduce the nature of $\sum_n u_n$.

Exercise 3

For any real number a, consider M_a as the matrix in $\mathcal{M}_3(\mathbb{R})$ defined by $M_a = \begin{pmatrix} 1 & 0 & 4 \\ a & 2 & 0 \\ 2 & 0 & 3 \end{pmatrix}$.

Part I :

1. Show that $u = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector of M_a, specifying the associated eigenvalue.
2. Show that for any value of $a \in \mathbb{R}$, the matrix M_a is diagonalisable.
3. Are there values of a for which the matrix M_a is invertible ?
4. We now seek functions x, y, and z of class $\mathcal{C}^1$ on $\mathbb{R}$ that satisfy the following system :

$$(\mathcal{S}) \quad \begin{cases} x'(t) = x(t) + 4z(t) \\ y'(t) = ax(t) + 2y(t) \\ z'(t) = 2x(t) + 3z(t) \end{cases}$$

   and the initial conditions $x(0) = 3$, $y(0) = 1$, and $z(0) = 0$. We define $X(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$ for all $t \in \mathbb{R}$. Notice that system $(\mathcal{S})$ can be rewritten as $X'(t) = M_a X(t)$.

   (a) Find a diagonal matrix D_1 and an invertible matrix P such that $M_a = PD_1P^{-1}$. It is not necessary to determine P^{-1}.

   (b) Let $Y(t) = P^{-1}X(t)$ with $Y(t) = \begin{pmatrix} \alpha(t) \\ \beta(t) \\ \gamma(t) \end{pmatrix}$. Show that $X'(t) = M_a X(t)$ if and only if $Y'(t) = D_1 Y(t)$.

   (c) Solve the system $Y'(t) = D_1 Y(t)$.

   (d) Determine $X(t)$ using the initial conditions.

Part II : We consider two real sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ satisfying the following system :

$$\begin{cases} u_{n+1} = u_n + 4v_n \\ v_{n+1} = 2u_n + 3v_n \end{cases}$$

with $u_0, v_0 \in \mathbb{R}$. Moreover, we define $U_n = \begin{pmatrix} u_n \\ v_n \end{pmatrix}$ for all $n \in \mathbb{N}$.

1. Determine the matrix N such that $U_{n+1} = NU_n$ for all $n \in \mathbb{N}$.
2. Express U_n in terms of U_0, N, and n.
3. (a) Determine a diagonal matrix D_2 and an invertible matrix Q such that $N = QD_2Q^{-1}$, then compute Q^{-1}.

   (b) Show that $N^n = QD_2^nQ^{-1}$ for all $n \in \mathbb{N}$.

   (c) Deduce u_n and v_n as functions of n, u_0, and v_0.
4. We now define $V_n = \begin{pmatrix} u_n \\ w_n \\ v_n \end{pmatrix}$ for all $n \in \mathbb{N}$. Using the previous questions, determine the expressions of the general terms of the sequences $(u_n)_{n\in\mathbb{N}}$, $(w_n)_{n\in\mathbb{N}}$, and $(v_n)_{n\in\mathbb{N}}$ satisfying the relation $V_{n+1} = M_0V_n$, in terms of n, u_0, v_0, and w_0.

Exercice 4

We recall that $1 + 2 + \ldots + n = \dfrac{n(n+1)}{2}$.
Let $n \in \mathbb{N}^\star$. For all $x \in \mathbb{R}$, we define

$$A(x) = \begin{pmatrix} 1+x & 1 & \ldots & 1 \\ 2 & 2+x & \ldots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ n & n & \ldots & n+x \end{pmatrix}.$$

1. Compute $\det(A(x))$.
2. We aim to retrieve the result from the previous question by directly evaluating the eigenvalues of $A(x)$.
   (a) Determine the rank of $A(0)$.
   (b) Deduce that for all $x \in \mathbb{R}$, $A(x)$ has an eigenvalue with multiplicity at least $n-1$, which depends on x and will be specified.
   (c) Show that $\lambda = x + \dfrac{n(n+1)}{2}$ is an eigenvalue of $A(x)$.
   (d) Deduce from (b) and (c) the value of $\det(A(x))$.